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QUOTIENTS OF QUATERNIONIC HOLOMORPHIC SECTIONS

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Abstract. A surface is represented as a quotient of two quaternionic holomorphic sections. Utilizing these quotients, we explain a correspondence between super-conformal surfaces and complex holomorphic null curves.

1. Introduction

The theory of quaternionic analysis on a Riemann surface by Pedit and Pinkall [3] draws an analogy between complex holomorphic functions on a Riemann surface and weakly-conformal immersions from a Riemann surface to the Euclidean four-space. A complex holomorphic section of a trivial complex line bundle over a Riemann surface is a holomorphic function. A quaternionic holomorphic section of a trivial complex quaternionic line bundle over a Riemann surface is a weakly-conformal immersion from a Riemann surface to the Euclidean four-space.

A quotient of two complex holomorphic sections of a complex line bundle is a complex holomorphic function on a Riemann surface. Similarly, a quotient of two quaternionic holomorphic sections of a complex quaternionic line bundle is a weakly-conformal immersion from a Riemann surface by [3, p. 395, Example].

When we consider holomorphic one-forms, we have a similar situation. A quotient of two complex holomorphic one-forms is a complex holomorphic function on a Riemann surface. A quotient of quaternionic holomorphic one-forms is a weakly-conformal immersion. We will review quotients of quaternionic holomorphic one-forms and updated the correspondence between super-conformal surfaces and complex holomorphic null curves obtained in [4].

2. Surfaces and quaternions

In this section, we review quaternionic analysis on a Riemann surface by Pedit and Pinkall [3].

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2.1. **Conformal structures.** Let N be the Riemannian manifold and g_0 the Riemannian metric of N. We consider the conformal class c of Riemannian metrics on N where g_0 belongs to:

$$c = \{g = \lambda g_0 \mid \lambda \colon N \to \mathbb{R},\ \lambda > 0\}.$$

We denote by $\mathcal{C}$ the set of diffeomorphisms from N to N preserving the conformal structure c. The set $\mathcal{C}$ becomes a Lie group in a standard way. An action of $\mathcal{C}$ on N is naturally defined. Then $\mathcal{C}$ becomes a Lie transformation group of N. An element of $\mathcal{C}$ is called a conformal transformation of N.

2.2. **Surfaces.** We recall the description of surfaces in terms of quaternions.

Let E^4 be the Euclidean four-space, g_0 the Riemannian metric of E^4 and c the conformal structure of E^4 represented by g_0. We consider a two-dimensional oriented manifold M and an immersion $f \colon M \to E^4$. Then a conformal structure of M is induced from c by f. The conformal structure c and the orientation of M determines a complex structure of M such that $(v, J^M v)$ is a positive orthogonal basis of a tangent space of M for every non-zero tangent vector v of M.

We relax the condition for f. Let $M = (M, J^M)$ be a Riemann surface with complex structure J^M. We assume that $f \colon (M, J^M) \to E^4$ is a branched immersion such that J^M is orthogonal with respect to the metric induced from g_0 by f at every immersed point. Then f is a weakly-conformal immersion. We call f a surface.

Let $\mathbb{H}$ be the quaternions. The Euclidean four-space E^4 is identified with $\mathbb{H}$ in a natural manner. Then $f \colon (M, J^M) \to \mathbb{H}$ is an immersed surface if and only if there exist smooth maps $N,\ R \colon M \to \operatorname{Im}\mathbb{H}$ such that $*df := df \circ J^M = N\,df = -df\,R$. We see that $N^2 = R^2 = -1$. The maps N and R are called the left normal vector and the right normal vector of f. If f is branched, then N and R are not defined at branch points.

If $N = i$, then f is a complex holomorphic map from M to $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$. Similarly, If $R = -i$, then f is a complex holomorphic map from M to $\mathbb{C}^2 \cong \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{H}$. The map $f \colon M \to \mathbb{C} \subset \mathbb{H}$ is a complex holomorphic function if and only if $N = -R = i$. Hence we can consider a surface in $\mathbb{H}$ as an analogue of a complex holomorphic function.

2.3. **Quaternionic holomorphic structures of surfaces.** We introduce the terminology of vector bundles and consider equations which characterize holomorphic sections.

For a vector bundle V over a Riemann surface, we denote by $\Omega^n(V)$ the set of smooth sections of $V \otimes \bigwedge^n T^*M$ $(n = 0, 1, 2)$.

Let H be the right trivial quaternionic line bundle over M. A smooth map $\phi \colon M \to \mathbb{H}$ is considered as a smooth section of H. We fix a smooth map $N \colon M \to \operatorname{Im}\mathbb{H}$ with $N^2 = -1$. We define $D^N \colon \Omega^0(H) \to \Omega^1(H)$ by

$$D^N \phi = \frac{1}{2}\left(d\phi + N * d\phi\right).$$

If $D^N\phi = 0$, then ϕ is a constant map or a surface with left normal vector N. We call D^N the quaternionic holomorphic structure of a surface with left normal vector N. We see that

$$D^N(\phi\lambda) = (D^N\phi)\lambda + \frac{1}{2}(\phi\, d\lambda + N\phi * d\lambda) = (D^N\phi)\lambda + \phi D^{\phi^{-1}N\phi}\lambda$$
$$*D^N\phi = -ND^N\phi$$

for every $\lambda\colon M \to \mathbb{H}$.

Let C be the trivial complex line bundle over M. If $N = i$, then $D^i|_{\Omega^0(C)}$ is a complex holomorphic structure of C. Indeed,

$$D^i\phi = \frac{1}{2}\,(d\phi + i * d\phi) = \bar{\partial}\phi$$

for every $\phi \in \Omega^0(C)$. If $\bar{\partial}\phi = 0$, then ϕ is a complex holomorphic function. We have

$$\bar{\partial}(\phi\lambda) = \bar{\partial}\lambda + \frac{1}{2}(\phi\, d\lambda + i\phi * d\lambda) = (\bar{\partial}\phi)\lambda + \phi\,\bar{\partial}\lambda, \;\; *\,\bar{\partial}\phi = -i\,\bar{\partial}\phi$$

for every $\phi \in \Omega^0(C)$ and every $\lambda\colon M \to \mathbb{C}$.

3. Quotients of holomorphic sections

We review quotients of quaternionic holomorphic sections and update the correspondence between super-conformal surfaces and complex holomorphic null curves.

3.1. **Quotients of surfaces.** We assume that $\phi \in \Omega^0(H)$ is nowhere vanishing and $\lambda\colon M \to \mathbb{H}$ is a smooth map. If ϕ and $\phi\lambda$ are surfaces with left normal vector N, then the map $\lambda\colon M \to \mathbb{H}$ is a surface with left normal vector $\phi^{-1}N\phi$. Indeed, $D^N(\phi\lambda) = \phi D^{\phi^{-1}N\phi}\lambda = 0$. Since $\lambda = \phi^{-1}(\phi\lambda)$, we may say that a quotient of two surfaces with the same left normal vector is a surface.

This is an analogue of the fact that if ξ and η are complex holomorphic functions, then $\xi^{-1}\eta$ is a complex holomorphic functions.

3.2. **Quotients of quaternionic holomorphic one-forms.** We say that a quaternionic-valued one-form ω is quaternionic holomorphic with respect to N if $d\omega = 0$ and $*\omega = N\omega$. If ω is nowhere vanishing, then there exists a map $R\colon M \to \operatorname{Im}\mathbb{H}$ such that $N\omega = -\omega R$. By the definition, $R^2 = -1$. We assume that ω is a nowhere-vanishing, quaternionic holomorphic one-form with respect to N such that $*\omega = N\omega = -\omega R$. Let $\lambda\colon M \to \mathbb{H}$ be a smooth branched immersion such that $\omega\lambda$ is closed. Then

$$d(\omega\lambda) = -\omega \wedge d\lambda = 0.$$

In a similar way to the proof of Proposition 16 in [1], we see that λ is a surface with its left normal vector $-R$.

If ω is a complex holomorphic one-form, then $N = -R = i$. Let $\lambda\colon M \to \mathbb{C}$ be a smooth map such that $\omega\lambda$ is closed. Then $d(\omega\lambda) = -\omega \wedge d\lambda = 0$. This shows that $d\lambda$ is a complex holomorphic one-form. Hence λ is a complex holomorphic function.

3.3. **The Weierstrass representation.** Let N, $R\colon M \to \operatorname{Im}\mathbb{H}$ be maps which are the left normal vector and the right normal vector of a surface $f\colon (M, J^M) \to \mathbb{H}$ respectively. Let λ be a smooth branched immersion. If $df\,\lambda$ is closed, then λ is a surface with left normal vector $-R$. If $df\,\lambda$ is exact, then there exists a surface $g\colon (M, J^M) \to \mathbb{H}$ with left normal vector N such that $dg = df\,\lambda$. The equation $dg = df\,\lambda$ is considered as a Weierstrass representation of g. The relation between left normal vectors and right normal vectors are listed in Table 1.

	left normal	right normal
f	N_f	R_f
g	N_f	$\lambda^{-1}R_f\lambda$
λ	$-R_f$	R_λ

Table 1. Left normal vectors and right normal vectors

3.4. **Super-conformal surfaces and complex holomorphic null curves.** We recall a characterization of super-conformal surfaces.

Let $f\colon (M, J^M) \to \mathbb{H}$ be a surface. We assume that there exist maps N, $R\colon M \to \mathbb{H}$ such that N and R are the left normal vector and the right normal vector of f respectively. If $*dN - N\,dN = 0$ or $*dR - R\,dR = 0$, then f is super-conformal by [1, Theorem 5]. When we replace N and R to $-N$ and $-R$ respectively, then we have the equations $*dN + N\,dN = 0$ and $*dR + R\,dR = 0$. If $*dN + N\,dN = 0$ or $*dR + R\,dR = 0$, then f is minimal by [1, Proposition 8]. The equation $*dN + N\,dN = 0$ implies the equation $*dR + R\,dR = 0$ and vise versa. Hence, if the left normal vector of a minimal surface $f\colon (M, J^M) \to \mathbb{H}$ is the same as that of a minimal surface $g\colon (M, J^M) \to \mathbb{H}$, then the quaternionic-valued function λ defined by $dg = df\,\lambda$ is a super-conformal surface by Table 1.

A combination of two minimal surfaces is a complex holomorphic map. A complex holomorphic curve $\psi\colon M \to \mathbb{C}^4$ is called null if $\sum_{n=0}^{3} \partial\psi_n \otimes \partial\psi_n = 0$. We identify C^4 with $\mathbb{C} \otimes H$. Let $f = \operatorname{Re}\psi\colon M \to \mathbb{H}$ and $g = \operatorname{Im}\psi\colon M \to \mathbb{H}$. Then ψ is null if and only if f and g are minimal surfaces such that $*df = -dg$. The minimal surface f has the same left normal vector and the same right normal vector as the minimal surface g has.

Let $g_0 + ig_1\colon M \to \mathbb{C}^4$ be a complex holomorphic null curve with minimal surfaces g_0 and g_1. Then the map λ defined by $dg_1 = dg_0\,\lambda$ is a super-conformal surface. It is not trivial whether we can construct a complex holomorphic null cuve from a given super-conformal surface. The following theorem is an answer to this problem.

Theorem 1. *Let $N\colon M \to \operatorname{Im}\mathbb{H}$ be a map which is the left normal vector of a super-conformal surface $f\colon M \to \mathbb{H}$. We define a map $g_0\colon M \setminus \{p \,|\, (dN)_p = 0\} \to \mathbb{H}$ and $g_1\colon M \setminus \{p \,|\, (dN)_p = 0\} \to \mathbb{H}$ by $df = dN\,g_0$ and $g_1 = Ng_0 - f$. Then $g_0 + g_1 i\colon M \setminus \{p \,|\, (dN)_p = 0\} \to \mathbb{C}^4$ is a complex holomorphic null curve such that $*dg_0 = -N\,dg_0$.*

Conversely, let $g_0 + ig_1\colon M \to \mathbb{C}^4$ be a complex holomorphic null curve such that $*dg_0 = -N\,dg_0$. Then $f = Ng_0 - g_1\colon M \to \mathbb{H}$ is a super-conformal surface with $*df = N\,df$.

This is a variant of [4, Theorem 1]. Similar result is obtained in Dajczer and Tojeiro [2].

Proof. Let $f\colon M \to \mathbb{H}$ be a super-conformal surface with $*df = N\,df$. Then N is a surface with left normal vector N and right normal vector $-N$. A quaternionic-valued function g_0 defined by $df = dN\,g_0$ is a minimal surface with left normal vector $-N$. The domain of g_0 is $M \setminus \{p \,|\, (dN)_p = 0\}$. Since $dg_1 = N\,dg_0 = - * dg_0$, the map g_1 is a minimal surface and the map $g_0 + g_1 i$ is a complex holomorphic null curve.

Let $g_0 + g_1 i\colon M \to \mathbb{C}^4$ be a complex holomorphic null curve such that $*dg_0 = -N\,dg_0$. Since $*dg_0 = -dg_1$, we have $df = dN\,g_0 + N\,dg_0 - dg_1 = dN\,g_0$. The map satisfies the equation $*dN = N\,dN$. Hence f is a surface with left normal vector N. Hence f is a super-conformal surface. □

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