Proceedings of the 16th OCU
International Academic Symposium 2008
OCAMI Studies Volume 3
2009, pp.137–148

A CONSTRUCTION OF A FROBENIUS MANIFOLD FROM THE QUANTUM DIFFERENTIAL EQUATION OF A WEIGHTED PROJECTIVE SPACE

HIRONORI SAKAI

Abstract. Starting from the quantum differential equation associated to a weighted projective space, which is given by Coates, Corti, Lee and Tseng, we construct a Frobenius manifold. We see that the Frobenius manifold coincides with the big quantum cohomology of the weighted projective space. The construction is based on Dubrovin's reconstruction theorem.

Introduction

(Orbifold) Gromov-Witten invariants are defined on any closed symplectic orbifold [2]. The invariants give us two concepts playing central roles in mirror symmetry. One is a Frobenius manifold called the (big) quantum (orbifold) cohomology, and another is a system of differential equations called the quantum differential equation.

For closed symplectic toric manifold, the quantum differential equations are computed without explicit Gromov-Witten invariants [5, 9]. It is important to explore a way to compute Gromov-Witten invariants from the quantum differential equation, because it is very difficult to calculate Gromov-Witten invariants explicitly. Regarding this problem, Guest clarifies the relation between the quantum differential equations and the Birkhoff factorization [6], and the idea is successful for a wide range of smooth symplectic manifolds [1, 9, 12, 11].

For weighted projective spaces, Coates, Corti, Lee and Tseng calculate the quantum differential equation [3]. We studied the quantum differential equation associated to a weighted projective space from the point of view of the Birkhoff factorization in [8], and showed that the correct orbifold Gromov-Witten invariants may be extracted.

The purpose of this paper is to construct a Frobenius manifold directly from the quantum differential equation associated to a weighted projective space so that the Frobenius manifold is isomorphic to the big quantum cohomology. For the construction, we use Dubrovin's reconstruction theorem. Dubrovin's theorem says that a certain initial condition uniquely determines (the germ of) the Frobenius manifold [4]. Using only the quantum differential equation, we give the initial condition. Then Dubrovin's theorem guarantees the existence of a unique Frobenius manifold having the initial condition. We

Date: Received on January 2, 2010.
2000 *Mathematics Subject Classification.* Primary 53D45 Secondary 14N35.
Key words and phrases. Quantum cohomology, Orbifolds, D-modules.
This research is partially supported by the NSC grant 98-2115-M-006-006-MY2, and the NCTS (South).

call the Frobenius manifold the abstract quantum cohomology. Next we compute the initial data of the (usual) quantum cohomology. Applying Dubrovin's theorem again, we can conclude that the Frobenius manifold we construct coincides with the quantum cohomology of the weighted projective space. Our discussion is analogous to Mann's discussion [10]. There are two significant differences. One is that we compare the big quantum cohomology to the Frobenius manifold which we construct instead of the Frobenius manifold attached to a Laurent polynomial. Another is that we directly construct a Frobenius manifold from the quantum differential equation.

This paper is organized as follows. In §1, we review the weighted projective spaces and their quantum cohomology. Some notations we use later appear in the beginning of §1. Afterwords we define a weighted projective space and its orbifold cohomology. A weighted projective space is defined as an action groupoid and the orbifold cohomology of the weighted projective space is given by the ordinary cohomology of its inertia orbifold as additive space. The orbifold cohomology is equipped with a grading with values in rational numbers. In §2, we recall the quantum (orbifold) cohomology and the quantum differential equation associated to a weighted projective space.

Starting from the quantum differential equation, we construct a Frobenius manifold in §3. In order to give an initial data of the Frobenius manifold, we discuss a D-module associated to the quantum differential equation and its grading. Then Dubrovin's theorem gives a unique Frobenius manifold associated to the initial data. In §4, we compute the initial condition of the quantum cohomology of weighted projective spaces. The computation is essentially same as that of Mann [10]. Then we eventually conclude that these Frobenius manifolds are the same.

1. Quantum orbifold cohomology of weighted projective spaces

1.1. **Some notation.** For $n+1$ positive integers $w_0, w_1, \dots, w_n$, let

$$F := \left\{ \left[\frac{l}{w_j} \right] \in \mathbb{Q}/\mathbb{Z} \;\middle|\; l \in \mathbb{Z},\ 0 \le j \le n \right\} = \big\{ [f_1], \dots, [f_k] \big\}$$

where $f_1, \dots, f_k$ is the sequence of fractions satisfying $0 = f_1 < f_2 < \cdots < f_k < 1$. We set $f_{k+1} := 1$ for convenience. The "multiplicity" of the fraction f_μ is denoted by u_i. The multiplicity u_i can also be calculated as the cardinality of the set $S_\mu := \{ i \mid f_\mu w_i \in \mathbb{Z} \}$. Put

$$s = u_1 + \cdots + u_k = w_0 + \cdots + w_n.$$

It is easy to see the following lemma.

Lemma 1 (Lemma 2.1 in [8])**.**

(1) $f_\mu + f_\nu = 1$ *if and only if* $\mu + \nu = k + 2$.
(2) $u_\mu = u_\nu$ *if* $\mu + \nu = k + 2$.
(3) $u_2 + \cdots + u_\mu = u_{k+2-\mu} + \cdots + u_k$ *for* $2 \le \mu \le k$.

The weighted projective space can be described as a groupoid. A Lie groupoid is a category $\mathcal{G}$ satisfying the following conditions:

(1) The class $\mathcal{G}_0$ of objects and the class $\mathcal{G}_1$ of arrows both are manifolds.
(2) Every arrow is invertible.

(3) All of structure maps are smooth. The structure maps are the five maps following below.
- The source map $s : \mathcal{G}_1 \to \mathcal{G}_0;\ (g : x \to y) \mapsto x$.
- The target map $t : \mathcal{G}_1 \to \mathcal{G}_0;\ (g : x \to y) \mapsto y$.
- The composition $c : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1;\ (h : y \to z, g : x \to y) \mapsto (hg : x \to z)$.
- The unit map $u : \mathcal{G}_0 \to \mathcal{G}_1;\ x \mapsto \mathrm{id}_x$.
- The inverse map $t : \mathcal{G}_1 \to \mathcal{G}_1;\ (g : x \to y) \mapsto (g^{-1} : y \to x)$.

Here $x, y \in \mathcal{G}_0$ and $g \in \mathrm{Hom}(x, y) \subset \mathcal{G}_1$.

(4) the the source map s and the target map t both are surjective submersions.

For $x, y \in \mathcal{G}_0$ we say $x \sim y$ if there exists an arrow $g : x \to y$ connecting between x and y. The relation $\sim$ is an equivalence relation. The underlying space $|\mathcal{G}|$ of the Lie groupoid $\mathcal{G}$ is the quotient topological space of $\mathcal{G}_0$ under the equivalence relation $\sim$.

If a Lie group G acts on a manifold M on the left, the action groupoid $G \ltimes M$ is defined as follows. The space $(G \ltimes M)_0$ of objects is M and the space $(G \ltimes M)_1$ of arrows is $G \times M$. The source map $s : G \times M \to M$ is the projection and the target map $t : G \times M \to M$ is the action map. The unit map is given by $u : x \mapsto (e, x)$, where e is the identity of G. The inverse map is defined by $i : (g, x) \mapsto (g^{-1}, g \cdot x)$. Any action groupoid is a Lie groupoid.

1.2. **The weighted projective spaces.** Let $\mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}$ be the 1-dimensional torus. The weighted projective space $\mathbb{P}(\boldsymbol{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is the action groupoid $\mathbb{T} \ltimes S^{2n+1}$, where S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ and the torus $\mathbb{T}$ acts on the sphere S^{2n+1} with negative weights $-w_0, -w_1, \ldots, -w_n$:

$$\mathbb{T} \curvearrowright S^{2n+1};\ t \cdot (z_0, z_1, \ldots, z_n) = (t^{-w_0} z_0, t^{-w_1} z_1, \ldots, t^{-w_n} z_n).$$

Let $\mathcal{I}$ be the subset of $\mathbb{T}$ consisting of elements having fixed points. There is a bijection between F and $\mathcal{I}$: $F \to \mathcal{I};\ [f] \mapsto e^{2\pi \mathtt{i} f}$, where $\mathtt{i} = \sqrt{-1}$. Therefore we identify $\mathcal{I}$ with F. If there is no confusion, an element of F is written as f instead of $[f]$ and we assume f to satisfy $0 \le f \le 1$.

We can associate the inertia orbifold to each (almost complex) orbifold. The *inertia orbifold* $\wedge\mathbb{P}(\boldsymbol{w})$ is the action groupoid $\mathbb{P}(\boldsymbol{w}) \ltimes S_{\mathbb{P}(\boldsymbol{w})}$, where

$$S_{\mathbb{P}(\boldsymbol{w})} := \left\{(t, z) \in \mathbb{T} \times S^{2n+1} \,\middle|\, t \cdot z = z\right\}.$$

Since the action of $\mathbb{P}(\boldsymbol{w})$ on $S_{\mathbb{P}(\boldsymbol{w})}$ is naturally identified with the action of the torus $\mathbb{T}$, we have

$$\wedge\mathbb{P}(\boldsymbol{w}) = \bigsqcup_{f \in F} \mathbb{T} \ltimes \left(S^{2n+1} \cap V_f\right) = \bigsqcup_{f \in F} \mathbb{P}(\boldsymbol{w})_f$$

where $V_f := \left\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, z_i = 0 \text{ if } f w_i \notin \mathbb{Z}\right\}$ and $\mathbb{P}(\boldsymbol{w})_f$ is the action groupoid $\mathbb{T} \ltimes \left(S^{2n+1} \cap V_f\right)$. Thus we can think of F as the index set of the components of $\mathbb{P}(\boldsymbol{w})$. The complex dimension of $\mathbb{P}(\boldsymbol{w})_{f_\mu}$ is equal to $u_\mu + 1$. Note that $\mathbb{P}(\boldsymbol{w})_0$ is nothing but $\mathbb{P}(\boldsymbol{w})$, i.e. the inertia orbifold $\wedge\mathbb{P}(\boldsymbol{w})$ has the original orbifold $\mathbb{P}(\boldsymbol{w})_0$ as a component. The component is called the *non-twisted sector* and another components are called *twisted sectors*. The involution of $S_{\mathbb{P}(\boldsymbol{w})}$, $(t, z) \mapsto (t^{-1}, z)$, induces an involution on $\wedge\mathbb{P}(\boldsymbol{w})$ and F. We denote by inv both involutions.

We associate a *degree shifting number* (or *age*) to each component of the inertia orbifold. For $z \in S^{2n+1}$ the local group at z is naturally identified with the group

$\mathbb{T}_z$ of stabilizers of z. Since the stabilizer is finite, the action of the local group $\mathbb{T}_z$ on the tangent space $T_z\mathbb{P}(\boldsymbol{w})$ can be decomposed into a direct sum of eigenspaces: $T_z\mathbb{P}(\boldsymbol{w}) = \bigoplus_{0\le f<1} T(f)$. Here $T(f)$ is the $e^{2\pi\mathtt{i}f}$-eigenspace. The degree shifting number is defined by

$$\iota_f := \sum_{0\le f<1} f \dim_{\mathbb{C}} T(f).$$

The degree shifting number depends only on the choice of a component of $\wedge\mathbb{P}(\boldsymbol{w})$.

The *orbifold cohomology* (or *Chen-Ruan cohomology*) is defined by

$$\mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w});\mathbb{C}) := \bigoplus_{f\in F} \mathrm{H}^{*-2\iota_f}(\mathbb{P}(\boldsymbol{w})_f;\mathbb{C}).$$

The orbifold cohomology is the same as the ordinary cohomology of the inertia orbifold as an additive vector space, but the degree is different. For example, a homogeneous cohomology class $x \in H^k(\mathbb{P}(\boldsymbol{w})_f;\mathbb{C})$ has degree $k + 2\iota_f$. It is called the *orbifold degree* of x and denoted by $\deg x$.

The orbifold cohomology is equipped with a bilinear form, which is called the *orbifold Poincaré pairing*, defined by

$$(x, y)_{\mathrm{orb}} = \int_{\wedge\mathbb{P}(\boldsymbol{w})} x \cup \mathrm{inv}^* y.$$

Here $\cup$ is the usual cup product of the ordinary cohomology of the inertia orbifold. It is known that the pairing is nondegenerate because of the compactness of $\mathbb{P}(\boldsymbol{w})$.

2. Quantum cohomology

A *Frobenius manifold* is a quadruple $(M, g, \circ, e)$ consisting of the following objects.

- M is a complex manifold.
- g is a $\mathbb{C}$-bilinear form on the holomorphic tangent bundle TM. We call g a *metric*.
- $\circ : TM \otimes TM \to TM$ is a symmetric multiplication.
- e is a vector field satisfies the identity $e \circ X = X$ for all vector field X, which is called an *identity*.

We assume the following properties:

(1) The metric g is flat, i.e. its Levi-Civita connection ∇^{LC} has zero curvature. Note that the Levi-Civita connection is well-defined in the usual sense.
(2) The Frobenius condition $g(X \circ Y, Z) = g(X, Y \circ Z)$ holds for all vector fields X, Y, Z.
(3) The identity e is flat with respect to the Levi-Civita connection: $\nabla^{\mathrm{LC}} e = 0$.

An *Euler field* E of a Frobenius manifold $(M, g, \circ, e)$ is a global vector field of M such that there exist constants $D \in \mathbb{C}$ and identities

$$\mathcal{L}_E g = Dg \quad \text{i.e.} \quad E(g(X,Y)) - g([E,X],Y) - g(X,[E,Y]) = Dg(X,Y)$$

and

$$\mathcal{L}_E \circ = \circ \quad \text{i.e.} \quad [E, X\circ Y] - [E,X]\circ Y - X\circ[E,Y] = X\circ Y$$

holds for all vector fields X, Y. Here $\mathcal{L}_E$ is the Lie derivative along E.

The (genus zero) Gromov-Witten invariants are defined using integration over the virtual fundamental class of the moduli space of orbifold stable maps:

$$\Psi_{0,l,A}(\alpha_1,\dots,\alpha_l) := \int_{[\overline{\mathcal{M}}_{0,l}(\mathbb{P}(\boldsymbol{w}),A)]^{\mathrm{vir}}} \mathrm{ev}_1^*\,\alpha_1 \cup \dots \cup \mathrm{ev}_l^*\,\alpha_l.$$

Here $\overline{\mathcal{M}}_{0,l}(\mathbb{P}(\boldsymbol{w}),A)$ is the moduli space of orbifold stable maps of genus 0, with l marked points, and its homology class $A \in \mathrm{H}_2(|\mathbb{P}(\boldsymbol{w})|;\mathbb{Z})$, $|\mathbb{P}(\boldsymbol{w})|$ is the underlying space of $\mathbb{P}(\boldsymbol{w})$ and $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,l}(\mathbb{P}(\boldsymbol{w}),A) \to |\wedge\mathbb{P}(\boldsymbol{w})|$ is the ith evaluation map.

Let η_{f_μ} be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})_{f_\mu}}(1)$. Then

$$\mathbf{1}_{f_1}, \eta_{f_1}, \eta_{f_1}^2, \dots, \eta_{f_1}^{u_1-1}, \ \dots, \ \mathbf{1}_{f_k}, \eta_{f_k}, \eta_{f_k}^2, \dots, \eta_{f_k}^{u_k-1}$$

form a basis of $\mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w});\mathbb{C})$, where $\mathbf{1}_{f_\mu}$ is the unit cohomology class of the ordinary cohomology $\mathrm{H}^*(\mathbb{P}(\boldsymbol{w})_{f_\mu};\mathbb{C})$. The above basis is denoted by $p_0, p_1, \dots, p_{s-1}$.

Let $(t_0,\dots,t_{s-1})$ be the coordinate system of $\mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w});\mathbb{C})$ with respect to the basis $p_0,\dots,p_{s-1}$. The (big) quantum cup product $\bullet_\tau$ at $\tau = t_0p_0 + \dots + t_{s-1}p_{s-1}$ is defined by

$$(x \bullet_\tau y, z)_{\mathrm{orb}} = \sum_{m\geq 0} \sum_{A\in \mathrm{H}_2(|\mathbb{P}(\boldsymbol{w})|;\mathbb{Z})} \frac{1}{m!}\Psi_{0,m+3,A}(x,y,z,\underbrace{\tau,\dots,\tau}_{m\text{times}})$$

for all $z \in \mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w});\mathbb{C})$. The convergence of the big quantum cup product might be assumed.

Assumption 1. We assume that there exists a small open neighbourhood M of $0 \in \mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w});\mathbb{C})$ so that the quantum cup product converges on M.

Proposition 1 (Chen-Ruan [2]). *The quadruple $(M, (\ ,\ )_{\mathrm{orb}}, \bullet, p_0)$ forms a Frobenius manifold. The Frobenius manifold is called the (big) quantum cohomology of $\mathbb{P}(\boldsymbol{w})$.*

An Euler field is given by

$$E = s\frac{\partial}{\partial t_1} + \sum_{\alpha=0}^{s-1}\left(1 - \frac{\deg p_\alpha}{2}\right) t_\alpha \frac{\partial}{\partial t_\alpha}.$$

From now on, we identify $\mathrm{H}^2(\mathbb{P}(\boldsymbol{w});\mathbb{C}) = \mathbb{C}p_1$ with $\mathbb{C}$ and the coordinate function t_1 is also denoted by t. Let $\mathcal{O}^\hbar$ be the sheaf of holomorphic functions in t with values in polynomials in $\hbar$ and $\mathcal{D}^\hbar := \mathcal{O}^\hbar[\hbar\partial]$ be the ring of differential operators generated by $\hbar\partial$ with coefficients in $\mathcal{O}^\hbar$. The restriction of $\mathcal{T}_M \otimes \mathcal{O}^\hbar$ to $\mathrm{H}^2(\mathbb{P}(\boldsymbol{w});\mathbb{C})$ is denoted by $\mathcal{E}_{\mathrm{qd}}$. The quantum cup product gives a $\mathcal{D}^\hbar$-module action on the sheaf $\mathcal{E}_{\mathrm{qd}}$ as follows:

$$\mathcal{D}^\hbar \curvearrowright \mathcal{E}_{\mathrm{qd}}; \quad \hbar\partial \cdot \xi := \hbar d\xi + p_1 \bullet_{tp_1} \xi \quad \big(\xi \in \mathcal{E}_{\mathrm{qd}}\big).$$

The $\mathcal{D}^\hbar$-module is called the *(small) quantum D-module* of $\mathbb{P}(\boldsymbol{w})$. Coates, Corti, Lee and Tseng calculate the annihilator of p_0.

Theorem 1 (Coates-Corti-Lee-Tseng [3]). *The annihilator of p_0 is the left ideal generated by $T_{\boldsymbol{w}} - e^t$, where*

$$T_{\boldsymbol{w}} := \prod_{i=0}^{n}\prod_{\mu=0}^{w_i-1}(w_i\hbar\partial - \mu h).$$

The differential equation given by the operator $T_{\boldsymbol{w}} - e^t$ is called the *quantum differential equation* associated to $\mathbb{P}(\boldsymbol{w})$.

3. A Frobenius manifold associated to the quantum differential equation

In this section, we construct a Frobenius manifold from quantum differential equation which will turn out to be equivalent to the big quantum cohomology.

Theorem 2 (Dubrovin [4]; see also Theorem 2.1 in Mann [10])**.** *Suppose we have the following data.*

(1) *A nondegenerate symmetric bilinear form* $g_0 : \mathbb{C}^s \times \mathbb{C}^s \to \mathbb{C}$ *on* $\mathbb{C}^s$*.*
(2) *A semisimple and invertible linear transformation* $A_0 : \mathbb{C}^s \to \mathbb{C}^s$ *which is self-adjoint with respect to the bilinear form* g_0*, i.e.* $A_0 = A_0^*$*. Here* A_0^* *is the adjoint operator of* A_0 *with respect to the bilinear form* g_0*.*
(3) *A linear transformation* $A_\infty : \mathbb{C}^s \to \mathbb{C}^s$ *such that* $A_\infty^* + A_\infty = k\,\mathrm{id}$ *for some integer* k*.*
(4) *An eigenvector* $e_0 \in \mathbb{C}^s$ *of* A_∞ *such that* $e_0, A_0e_0, A_0^2e_0, \dots, A_0^{s-1}e_0$ *form a basis of* $\mathbb{C}^s$*.*

Then there exists a unique germ of Frobenius manifold $(M, g, \circ, e)$ *with Euler field* E *such that* M *is an open neighbourhood of* $0 \in \mathbb{C}^s$ *and under the identification* T_0M *with* $\mathbb{C}^s$ *the following identities hold:*

$$g|_{\tau=0} = g_0, \quad (E\circ)|_{\tau=0} = A_0, \quad A_\infty = (\lambda+1)\,\mathrm{id} - \nabla E \quad \textit{and} \quad e|_{\tau=0} = e_0.$$

Here τ *is the parameter of* M *and* λ *is the eigenvalue of* e_0*. We call the quadruple* $(g_0, A_0, A_\infty, e_0)$ *an* initial condition *of a Frobenius manifold.*

3.1. **Abstract quantum cohomology.** Given the initial condition, we construct an object which we call abstract quantum cohomology. Define a $\mathcal{D}^\hbar$-module $\mathcal{E}_{\mathrm{abs}}$ as the quotient module $\mathcal{D}^\hbar/(T_{\boldsymbol{w}} - e^t)$. Recall the notation defined in Section 1.2. Define Δ_μ and m_μ as follows.

$$\Delta_\mu := f_{\mu+1} - f_\mu, \quad m_\mu := \prod_{i \in S_\mu} w_i \quad \text{for } 1 \le \mu \le k.$$

Using the identity $e^{-\frac{\mu}{w_i}t}(w_i\hbar\partial - \mu h) = w_i(\hbar\partial)e^{-\frac{\mu}{w_i}t}$ repeatedly, we obtain the following description [8]:

$$e^{-t}T_{\boldsymbol{w}} - 1 = \frac{1}{r_s}(\hbar\partial)\ \frac{1}{r_{s-1}}(\hbar\partial)\cdots\frac{1}{r_2}(\hbar\partial)\ \frac{1}{r_1}(\hbar\partial),$$

where

$$r_\alpha := \begin{cases} \dfrac{1}{m_\mu}e^{\Delta_\mu t} & \text{if } \alpha = u_1 + \cdots + u_\mu \text{ for some } \mu, \\ 1 & \text{else.} \end{cases}$$

Put $P_0 = 1$ and define P_α inductively by $P_\alpha = \dfrac{1}{r_\alpha}\hbar\partial \cdot P_{\alpha-1}$ for $\alpha \in \mathbb{Z}_{\ge 0}$. The equivalence class $[P_\alpha]$ of P_α in $\mathcal{E}_{\mathrm{abs}}$ is denoted by ξ_α. Then $\xi_0, \dots, \xi_{s-1}$ form an $\mathcal{O}^\hbar$-basis of $\mathcal{E}_{\mathrm{abs}}$. Consider the $\mathbb{C}$-vector space V generated by $\xi_0, \dots, \xi_{s-1}$: $V = \mathbb{C}\xi_0 \oplus \cdots \oplus \mathbb{C}\xi_{s-1} \cong \mathbb{C}^s$. We give data stated in Theorem 2 on V as follows.

(1) We define a bilinear form on $\mathcal{E}_{\mathrm{abs}}$ [8]. The key ingredient is the dual of the $\mathcal{D}^\hbar$-module. Define $\mathcal{E}_{\mathrm{abs}}^*$ by

$$\mathcal{E}_{\mathrm{abs}}^* = \mathrm{Hom}_{\mathcal{O}^\hbar}(\mathcal{E}_{\mathrm{abs}}, \mathcal{O}^\hbar).$$

The natural $\mathcal{O}^\hbar$-bilinear pairing between $\mathcal{E}_{\mathrm{abs}}^*$ and $\mathcal{E}_{\mathrm{abs}}$ is denoted by $\langle\ ,\ \rangle$. The sheaf $\mathcal{E}_{\mathrm{abs}}^*$ is equipped with the induced $\mathcal{D}^\hbar$-module action $\odot$ defined by

$$\mathcal{D}^\hbar \curvearrowright \mathcal{E}_{\mathrm{abs}}^*;\ \begin{cases} \langle \hbar\partial \odot \varphi, \xi\rangle = \langle \varphi, \hbar\partial \cdot \xi\rangle - \hbar\partial\langle\varphi,\xi\rangle, \\ \langle f(t,\hbar)\odot\varphi,\xi\rangle = \langle\varphi, f(t,-\hbar)\xi\rangle\ \big(= f(t,-\hbar)\langle\varphi,\xi\rangle\big), \end{cases}$$

where $\varphi \in \mathcal{E}_{\mathrm{abs}}^*$, $\xi \in \mathcal{E}_{\mathrm{abs}}$ and $f \in \mathcal{O}^\hbar$.

Let $\delta_0, \ldots, \delta_{s-1}$ be the dual basis for $\mathcal{E}_{\mathrm{abs}}^*$ to $\xi_0, \ldots, \xi_{s-1}$. The $\mathcal{D}^\hbar$-module action gives an explicit isomorphism:

$$\mathcal{E}_{\mathrm{abs}} \cong \mathcal{E}_{\mathrm{abs}}^*\ ;\quad [P] \leftrightarrow P \odot \delta_n.$$

The isomorphism induces the pairing on $\mathcal{E}_{\mathrm{abs}}$:

$$(\!(\ ,\ )\!) : \mathcal{E}_{\mathrm{abs}} \times \mathcal{E}_{\mathrm{abs}} \to \mathcal{O}^\hbar\ ;\quad (\!([P],[Q])\!) := \frac{1}{w_0 w_1 \ldots w_n}\langle P \odot \delta, [Q]\rangle.$$

The pairing is explicitly given by

$$(\!(\xi_\alpha, \xi_\beta)\!) = \begin{cases} \dfrac{1}{m_1}\delta_{n-\alpha,\beta} & \text{if } 0 \le \alpha < u_1, \\ \dfrac{1}{m_{\mu+1}}\delta_{s+n-\alpha,\beta} & \text{if } u_1 + \cdots + u_\mu \le \alpha < u_1 + \cdots + u_{\mu+1}, \end{cases}$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore the bilinear form is nondegenerate and symmetric. Let g_{abs} be the restriction of the bilinear form $(\!(\ )\!)$ to V.

(2) Define a linear transformation $A_{0,\mathrm{abs}}$ by

$$A_{0,\mathrm{abs}} : V \to V;\ \xi_\alpha \mapsto (s\hbar\partial \cdot \xi_\alpha)|_{t=0}.$$

Here "$|_{t=0}$" means to put $t = 0$ in each coefficients of ξ_β's. By the definition of the ξ_α's we have

$$\begin{aligned} A_{0,\mathrm{abs}}\xi_{\alpha-1} &= (sr_\alpha\xi_\alpha)|_{t=0} \\ &= \begin{cases} \dfrac{s}{m_\mu}\xi_\alpha & \text{if } \alpha = u_1 + \cdots + u_\mu \text{ for some } \mu, \\ s\xi_\alpha & \text{otherwise.} \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} g_{\mathrm{abs}}(A_{0,\mathrm{abs}}\xi_\alpha, \xi_\beta) &= (\!(s\hbar\partial\cdot\xi_\alpha, \xi_\beta)\!)|_{t=0} \\ &= (\!(\xi_\alpha, s\hbar\partial\cdot\xi_\beta)\!)|_{t=0} - s\hbar\partial\cdot(\!(\xi_\alpha,\xi_\beta)\!)|_{t=0} \\ &= g_{\mathrm{abs}}(\xi_\alpha, A_{0,\mathrm{abs}}\xi_\beta). \end{aligned}$$

Therefore the matrix $A_{0,\mathrm{abs}}$ is self-adjoint with respect to the metric g_{abs}.

(3) We introduce a subspace of $\mathcal{E}_{\mathrm{abs}}$ which has a grading structure (cf. Section 5.2 in the author's thesis [11]). We define a differential operator $E^\hbar$ by

$$E^\hbar := \hbar\frac{\partial}{\partial\hbar} + s\frac{\partial}{\partial t}.$$

so that $T_{\boldsymbol{w}} - e^t$ will be "homogeneous". For $\lambda \in \mathbb{Q}$, define the space $\mathcal{O}^\hbar(\lambda)$ of degree-λ functions by

$$\begin{aligned}\mathcal{O}^\hbar(\lambda) &:= \{f \in \mathcal{O}^\hbar \mid \mathcal{L}_{E^\hbar} f = \lambda f\} \\ &= \operatorname{Span}_{\mathbb{C}}\{\hbar^i e^{\alpha t} \mid i \in \mathbb{Z}_{\geq 0},\ \alpha \in \mathbb{Q},\ \lambda = i + s\alpha\}.\end{aligned}$$

Then $\mathcal{O}^\hbar(*) := \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}^\hbar(\lambda)$ is a graded ring.

Next we introduce a partial grading on $\mathcal{D}^\hbar$. For $f \in \mathcal{O}^\hbar$ and nonnegative integer j the action of $\mathcal{L}_{E^\hbar}$ on the operator $f(\hbar\partial)^j$ is defined by

$$\mathcal{L}_{E^\hbar} f(\hbar\partial)^j = (\mathcal{L}_{E^\hbar} f)(\hbar\partial)^j + f \cdot j(\hbar\partial)^j.$$

The space of degree-λ differential operators is denoted by $\mathcal{D}^\hbar(\lambda)$:

$$\begin{aligned}\mathcal{D}^\hbar(\lambda) &:= \{P \in \mathcal{D}^\hbar \mid \mathcal{L}_{E^\hbar} P = \lambda P\} \\ &= \left\{\textstyle\sum_j f_j(\hbar\partial)^j \in \mathcal{D}^\hbar \,\middle|\, f_j \in \mathcal{O}^\hbar(\lambda - j)\right\}.\end{aligned}$$

It is obvious that the direct sum $\mathcal{D}^\hbar(*) := \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{D}^\hbar(\lambda)$ is a graded module over $\mathcal{O}^\hbar(*)$. Moreover $\mathcal{D}^\hbar(*)$ is a graded ring.

It is trivial that the equivalence class $[1] \in \mathcal{E}_{\mathrm{abs}}$ including 1 is a cyclic generator of the $\mathcal{D}^\hbar$-module, i.e. $\mathcal{E}_{\mathrm{abs}} = \mathcal{D}^\hbar \cdot [1]$ and its annihilator $\mathcal{J}$ in $\mathcal{D}^\hbar(*)$ coincides with the left ideal generated by $T_{\boldsymbol{w}} - e^t$. Since a direct calculation implies that the operator $T_{\boldsymbol{w}} - e^t$ is homogeneous of degree s, the ideal $\mathcal{J}$ is homogeneous. Set $\mathcal{E}_{\mathrm{abs}}(\lambda) := \mathcal{D}^\hbar(\lambda) \cdot [1]$. If $\lambda_1 \neq \lambda_2$, $\mathcal{E}_{\mathrm{abs}}(\lambda_1) \cap \mathcal{E}_{\mathrm{abs}}(\lambda_2) = \{0\}$. The direct sum $\mathcal{E}_{\mathrm{abs}}(*) = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{E}_{\mathrm{abs}}(\lambda)$ is a graded over $\mathcal{D}^\hbar(*)$.

We say $\deg \xi = \lambda$ if $\xi \in \mathcal{E}_{\mathrm{abs}}(\lambda)$. In particular $\deg \xi_\alpha = \alpha$ if $0 \leq \alpha < u_1$. Moreover the definition of the r_α's implies that for α satisfying $u_1 + \cdots + u_\mu \leq \alpha < u_1 + \cdots + u_{\mu+1}$ we have

$$\deg \xi_\alpha = \alpha - (\Delta_1 s + \cdots + \Delta_\mu s) = \alpha - f_{\mu+1} s.$$

Define a linear map $A_{\infty,\mathrm{abs}} : \mathbb{C}^s \to \mathbb{C}^s$ as the grading operator:

$$A_{\infty,\mathrm{abs}} : V \to V;\ \xi_\alpha \mapsto (\deg \xi_\alpha)\xi_\alpha$$

We can show $A_{\infty,\mathrm{abs}} + A^*_{\infty,\mathrm{abs}} = n\,\mathrm{id}_V$ as follows:

$$\begin{aligned}&g_{\mathrm{abs}}\big((A_{\infty,\mathrm{abs}} + A^*_{\infty,\mathrm{abs}})\xi_\alpha, \xi_\beta\big) \\ &= g_{\mathrm{abs}}(A_{\infty,\mathrm{abs}}\xi_\alpha, \xi_\beta) + g_{\mathrm{abs}}(\xi_\alpha, A_{\infty,\mathrm{abs}}\xi_\beta) \\ &= (\deg \xi_\alpha + \deg \xi_\beta) g_{\mathrm{abs}}(\xi_\alpha, \xi_\beta)\end{aligned}$$

If $0 \leq \alpha < u_1$, $g_{\mathrm{abs}}(\xi_\alpha, \xi_\beta)$ vanishes unless $n - \alpha = \beta$. We have

$$\deg \xi_\alpha + \deg \xi_\beta = \alpha + \beta = n.$$

If $u_1 + \cdots + u_\mu \leq \alpha < u_1 + \cdots + u_{\mu+1}$ for some integer μ, $g_{\mathrm{abs}}(\xi_\alpha, \xi_\beta)$ vanishes unless $s + n - \alpha = \beta$. Then the unique integer ν satisfies that $u_1 + \cdots + u_\nu \leq$

$\beta < u_1 + \cdots + u_{\nu+1}$ is equal to $k - \mu$ because

$$
\begin{aligned}
& u_1 + \cdots + u_\mu \leq s + n - \beta < u_1 + \cdots + u_{\mu+1} \\
\iff & u_{\mu+2} + \cdots + u_k < \beta - n < u_{\mu+1} + \cdots + u_k \\
\iff & u_2 + \cdots + u_{k-\mu} < \beta - u_1 + 1 \leq u_2 + \cdots + u_{k-\mu+1} \\
\iff & u_1 + \cdots + u_{k-\mu} \leq \beta < u_1 + \cdots + u_{k-\mu+1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\deg \xi_\alpha + \deg \xi_\beta &= (\alpha - f_{\mu+1} s) + (\beta - f_{k-\mu+1} s) \\
&= s + n - (f_{\mu+1} + f_{k-\mu+1}) s \\
&= n.
\end{aligned}
$$

Summarizing the above observation, for any α and β we obtain

$$g_{\mathrm{abs}}\big((A_{\infty,\mathrm{abs}} + A^*_{\infty,\mathrm{abs}})\xi_\alpha, \xi_\beta\big) = g_{\mathrm{abs}}(n\xi_\alpha, \xi_\beta).$$

Thus we can conclude that $A_{\infty,\mathrm{abs}} + A^*_{\infty,\mathrm{abs}} = n\,\mathrm{id}_V$.

(4) We take ξ_0 as the fourth part of the data. It is obvious that ξ_0 is a 0-eigenvector of $A_{\infty,\mathrm{abs}}$. The definition of the map $A_{0,\mathrm{abs}}$ implies that $A^\alpha_{0,\mathrm{abs}}\xi_0$ is a nonzero constant multiple of ξ_α. Therefore $\xi_0, A_{0,\mathrm{abs}}\xi_1, \ldots, A^{s-1}_{0,\mathrm{abs}}\xi_0$ form a basis of V.

According to Theorem 2, the initial condition $(g_{\mathrm{abs}}, A_{0,\mathrm{abs}}, A_{\infty,\mathrm{abs}}, \xi_0)$ generates the unique Frobenius manifold. We call the Frobenius manifold the abstract quantum cohomology.

Remark 1. The above construction works for the quantum differential equation of part of Fano complete intersections in complex projective spaces as follows. The factorization of the differential operator $e^{-t}T_{\boldsymbol{w}} - 1$ must be replaced by Birkhoff factorization. Birkhoff factorization gives a unique basis which is called a normalized trivialization [11].

(1) The bilinear form is obtained in the same way as above [7].
(2) Let $\hat{\Omega}$ be the matrix of the action of ∂ with respect to the basis. The matrix A_0 is given by $\hat{\Omega}|_{t=0}$ if the matrix is invertible. For some cases, e.g. a complete intersection of quadratic hypersurface and cubic hypersurface in the 6-dimensional complex projective space, the matrix fails to be invertible.
(3) We take $\mathrm{diag}(0, 2, 4, \ldots)$ as the matrix A_∞.
(4) The fourth part of the data is given in the same way as above.

The construction does not extend a non-Fano complete intersection, e.g. a Calabi-Yau hypersurface, in a complex projective space because the matrix $\hat{\Omega}$ does not convergent at $t = 0$. The construction also fails if the second Betti number is greater than 1. The reason is that the fourth part of the data requires the existence of one operator which generates the vector space spanned by the normalized trivialization.

3.2. **The initial condition of the quantum cohomology.** To compare the abstract quantum cohomology with the quantum cohomology of $\mathbb{P}(\boldsymbol{w})$, we see the initial condition of the quantum cohomology. A similar calculation can be found in Mann [10]. We identify the orbifold cohomology $\mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w}); \mathbb{C})$ with $\mathbb{C}^s$ via the basis $p_0, \ldots, p_{s-1}$.

(1) Firstly a nondegenerate symmetric bilinear form g_{qc} is the orbifold Poincaré pairing. We have

$$p_\alpha = \begin{cases} \eta_{f_1}^{\alpha} & \text{if } 0 \le \alpha < u_1, \\ \eta_{f_{\mu+1}}^{\alpha-u_1-\cdots-u_\mu} & \text{if } u_1+\cdots+u_\mu \le \alpha < u_1+\cdots+u_{\mu+1}. \end{cases}$$

Therefore if $0 \le \alpha < u_1$, we have

$$\begin{aligned} g_{\mathrm{qc}}(p_\alpha, p_\beta) &= \int_{\wedge\mathbb{P}(\boldsymbol{w})} p_\alpha \cup \mathrm{inv}^* p_\beta = \begin{cases} \int_{\wedge\mathbb{P}(\boldsymbol{w})} \eta_{f_1}^{\alpha+\beta} & \text{if } 0 \le \beta < u_1 \\ 0 & \text{else} \end{cases} \\ &= \frac{1}{m_1}\delta_{n-\alpha,\beta}. \end{aligned}$$

Let ν be the integer satisfying $u_1+\cdots+u_\nu \le \beta < u_1+\cdots+u_{\nu+1}$. If $u_1+\cdots+u_\mu \le \alpha < u_1+\cdots+u_{\mu+1}$, then we have

$$\begin{aligned} g_{\mathrm{qc}}(p_\alpha, p_\beta) &= \int_{\wedge\mathbb{P}(\boldsymbol{w})} p_\alpha \cup \mathrm{inv}^* p_\beta \\ &= \int_{\wedge\mathbb{P}(\boldsymbol{w})} \eta_{f_{\mu+1}}^{\alpha-u_1-\cdots-u_\mu} \cup \eta_{1-f_{\nu+1}}^{\beta-u_1-\cdots-u_\nu}. \end{aligned}$$

The above integration vanishes unless $f_{\mu+1} = 1-f_{\nu+1}$, i.e. $\mu+\nu = k$ (cf. Lemmna 1). Then we have

$$\begin{aligned} & (\alpha - u_1 - \cdots - u_\mu) + (\beta - u_1 - \cdots - u_\nu) = u_{\mu+1} - 1 \\ \Longleftrightarrow\ & \alpha+\beta = u_1+\cdots+u_\mu + u_{k+2-\nu} + \cdots + u_k + u_{\mu+1} - 1 \\ \Longleftrightarrow\ & \alpha+\beta = u_1+\cdots+u_k + u_1 - 1 \qquad (\because\ \mu+\nu = k) \\ \Longleftrightarrow\ & \alpha+\beta = s+n. \end{aligned}$$

Therefore

$$g_{\mathrm{qc}}(p_\alpha, p_\beta) = \frac{1}{m_{\mu+1}}\delta_{s+n-\alpha,\beta}.$$

(2) Define the linear map $A_{0,\mathrm{qc}}$ by

$$A_{0,\mathrm{qc}} : \mathbb{C}^s \to \mathbb{C}^s;\ p_\alpha \mapsto s p_1 \bullet_0 p_\alpha.$$

According to Coates, Corti, Lee and Tseng [3], the product is explicitly given by

$$A_{0,\mathrm{qc}} p_\alpha := s r'_{\alpha+1} p_{\alpha+1},$$

where r'_α is the constant defined by

$$r'_\alpha = \begin{cases} \prod_{i=0}^{n} w_i^{-\lceil f_{\mu+1} w_i \rceil + \lceil f_\mu w_i \rceil} & \text{if } \alpha = u_1+\cdots+u_\mu \text{ for some } \nu, \\ 1 & \text{else.} \end{cases}$$

Here $\lceil\ \rceil$ is the ceiling function. Applying the following lemma, it is easy to see that r'_α is equal to $(r_\alpha)|_{t=0}$.

Lemma 2. *For $\nu = 0, \ldots, k-1$,*

$$\langle -f_{\nu+1} w_i \rangle - \langle -f_\nu w_i \rangle = \begin{cases} 1 - \Delta_\nu w_i & \textit{if } i \in S_\nu, \\ -\Delta_\nu w_i & \textit{else}, \end{cases}$$

holds.

Proof. Let $l_\nu := \max\{l \in \mathbb{Z} \,|\, l \leq -f_\nu w_i\}$. By the definition of the set F, for any i the natural numbers $0, 1, \ldots, w_i - 1$ appear in the sequence $f_1 w_i, f_2 w_i, \ldots, f_k w_i$. Thus there are only two possibilities.

(a) The case $l_\nu = l_{\nu+1} + 1$. This condition is equivalent to $f_\nu w_i \in \mathbb{Z}$, i.e $i \in S_\nu$. We have

$$\begin{aligned}\langle -f_{\nu+1} w_i \rangle - \langle -f_\nu w_i \rangle &= (-f_{\nu+1} w_i - l_{\nu+1}) - (-f_\nu w_i - l_\nu) \\ &= 1 - \Delta_\nu w_i.\end{aligned}$$

(b) The case $l_\nu = l_{\nu+1}$. This condition is equivalent to $f_\nu w_i \in \mathbb{Z}$, i.e $i \notin S_\nu$.

$$\begin{aligned}\langle -f_{\nu+1} w_i \rangle - \langle -f_\nu w_i \rangle &= (-f_{\nu+1} w_i - l_{\nu+1}) - (-f_\nu w_i - l_\nu) \\ &= -\Delta_\nu w_i.\end{aligned}$$

□

(3) Define the linear map $A_{\infty,\mathrm{qc}}$ by

$$A_{\infty,\mathrm{qc}} : \mathbb{C}^s \to \mathbb{C}^s;\ p_\alpha \mapsto \frac{1}{2}(\deg p_\alpha) p_\alpha.$$

By the definition of the orbifold degree and the p_α's, we have

$$\deg p_\alpha = \begin{cases} 2\alpha & \text{if } 0 \leq \alpha < u_1, \\ 2(\alpha - u_1 - \cdots - u_\mu) + 2\iota_{f_{\mu+1}} & \text{otherwise.} \end{cases}$$

Here μ is the integer which satisfy $u_1 + \cdots + u_\mu \leq \alpha < u_1 + \cdots + u_{\mu+1}$. In order to compare $A_{\infty,\mathrm{qc}}$ with $A_{\infty,\mathrm{abs}}$, we calculate the orbifold degree. The degree shifting number is explicitly given by

$$\iota_f = \sum_{i=0}^{n} \langle -f w_i \rangle.$$

(See Section 5.7 in the author's thesis [11] for example.) Here $\langle r \rangle := r - \max\{i \in \mathbb{Z} \mid i \leq r\}$. In particular $\iota_{f_1} = \iota_0 = 0$.

Lemma 3. *If $\mu = 0, \ldots, k-1$, the identity.*

$$\iota_{f_{\mu+1}} = u_1 + \cdots + u_\mu - f_{\mu+1} s$$

holds.

Proof. According to Lemma 2, we have

$$\iota_{f_{\nu+1}} - \iota_{f_\nu} = \sum_{i=0}^{n} (\langle -f_{\nu+1} w_i \rangle - \langle -f_\nu w_i \rangle) = \sum_{i=0}^{n} (-\Delta_\nu w_i) + \sharp S_\nu = u_\nu - \Delta_\nu s.$$

Therefore we obtain

$$\begin{aligned}\iota_{f_{\mu+1}} &= (u_\nu - \Delta_\mu s) + (u_{\nu-1} - \Delta_{\mu-1} s) + \cdot + (u_1 - \Delta_1 s) + \iota_{f_0} \\ &= u_1 + \cdots + u_\mu - (\Delta_\mu + \cdots + \Delta_1) s \\ &= u_1 + \cdots + u_\mu - f_{\mu+1} s.\end{aligned}$$

□

(4) We take p_0 as the fourth part of the data of the initial condition.

Theorem 3. *The abstract quantum cohomology is isomorphic to the quantum cohomology as a Frobenius manifold.*

Proof. The initial condition $(g_{\mathrm{qc}}, A_{0,\mathrm{qc}}, A_{\infty,\mathrm{qc}}, p_0)$ corresponds to $(g_{\mathrm{abs}}, A_{0,\mathrm{abs}}, A_{\infty,\mathrm{abs}}, \xi_0)$ under the identifications $\mathrm{H}^*_{\mathrm{orb}}(\mathbb{P}(\boldsymbol{w}); \mathbb{C}) \cong \mathbb{C}^s \cong V$. Therefore the theorem follows from Theorem 2. □

Acknowledgments. The author thanks Martin Guest for several valuable comments.

References

[1] A. Amarzaya and M. A. Guest, Gromov-Witten invariants of flag manifolds, via D-modules, J. London Math. Soc. 72 (2005), 121–136.
[2] W. Chen and Y. Ruan, Orbifold Gromov-Witten theory, In Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310 (2002), 25–85.
[3] T. Coates, A. Corti, Y.-P. Lee, and H.-H. Tseng, The quantum orbifold cohomology of weighted projective space, Acta Math. 202 no. 2 (2009), 139–193.
[4] B. Dubrovin, Geometry of 2D topological field theories, In Integrable systems and quantum groups (Montecatini Terme, 1993), volume (1620) of Lecture Notes in Math., pages 120348, Springer, Berlin, 1996.
[5] A. Givental A mirror theorem for toric complete intersections, In Topological field theory, primitive forms and related topics (Kyoto, 1996), volume 160 of Progr. Math., pages 141175, Birkhäuser, Boston, 1998.
[6] M. Guest, Quantum cohomology via D-modules, Topology 44 (2005), 263–281.
[7] M. Guest, From quantum cohomology to integrable systems, Oxford University Press, Oxford, 2008.
[8] M. Guest and H. Sakai, Orbifold quantum D-modules associated to weighted projective spaces, arXiv:0810.4236 (preprint), 2008.
[9] H. Iritani, Quantum D-modules and equivariant Floer theory for free loop spaces, Math. Z. 252 (2006), 577–622.
[10] E. Mann, Orbifold quantum cohomology of weighted projective spaces, J. Algenraic Geom. 17 (2008), 137–166.
[11] H. Sakai, Normalization of quantum D-modules, PhD thesis, Tokyo Metropolitan University, 2009.
[12] H. Sakai, Gromov-Witten invariants of Fano hypersurfaces, revisited, J. Geom. Phys. 58 (2008), 654–669.

Department of Mathematics, National Cheng Kung University, No.1, University Road, Tainan City 701, Taiwan
E-mail address: trans@blueskyproject.net