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TOWARD NEVANLINNA-GALOIS THEORY FOR ALGEBRAIC MINIMAL SURFACES

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The purpose of my lecture at OCU2008 was to announce my recent attempt on extending the classical Nevanlinna theory of holomorphic maps $f : \mathbb{C} \to \mathbb{P}^1$ to the Weierstrass data (g, ω) ($g : M \to \mathbb{P}^1$ being the Gauss map) of algebraic minimal surfaces.

In this note, I will explain what I want to do with some background and then outline the idea I have in mind at this stage toward the proof of Conjecture 1 stated below.

A complete minimal surface in $\mathbb{R}^3$ is said to be *pseudo-algebraic*, if its Weierstrass data are defined on a compact Riemann surface finitely many points removed $M = \overline{M}\backslash\{P_1, \ldots, P_n\}$ and extend meromorphically across punctures ([KKM], see also [G]). The punctured Riemann surface M on which the Weierstrass data are defined is called the *basic domain*. The difference between a complete minimal surface being *algebraic* or *pseudo-algebraic* lies in whether the period condition [P] is required or not. It is well known that the period condition is always a very hard obstacle against the attempt constructing algebraic minimal surfaces. We write the Weierstrass data of the minimal surface under consideration as (g, ω). Then $g : M \to \mathbb{P}^1$ is the Gauss map. It is then a fundamental question to ask what is $\mathbb{P}^1\backslash g(M)$.

For algebraic minimal surfaces, Osserman's classical result [O1, O2] (see also [KKM]) says that the Gauss map can omit at most 3 values and so far no algebraic minimal surfaces whose Gauss map omits 3 values are discovered. On the other hand, in the celebrated papers [F1, F2], Fujimoto established the value distribution theory of the Gauss map of complete minimal surfaces. Fujimoto proved that the totally ramified value number of the Gauss map of any complete minimal surface in $\mathbb{R}^3$ is at most 4 and therefore $D_g := \sharp(\mathbb{P}^1\backslash g(M))$ is also at most 4 :

$$D_g \leq \nu_g \leq 4 \ .$$

This estimate is best possible in the sense that there exists a complete minimal surface in $\mathbb{R}^3$ whose Gauss map omits just 4 values (e.g., the Voss surface, which is pseudo-algebraic). On the other hand, it seems to be very hard to relate Fujimoto's aproach and the period condition. Therefore, the motivation of our study was originally an attempt establishing the value distribution theory for **algebraic** minimal surfaces. In

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this direction, Kawakami [Ka1] first found an example among algebraic minimal surfaces constructed by Miyaoka-Sato [MS], which satisfies the property

$$\nu_g = 2.5 \ ,$$

where ν_g is the totally ramified number of the Gauss map $g : M \to \mathbb{P}^1$. This observation is striking, because, if we believe that the methods in [F1, F2] could somehow be generalized to cover algebraic minimal surfaces, the expected result would just become "$\nu_g \le 2$" (if we believe that there is no algebraic minimal surface whose Gauss map omits 3 values) as was the case of general complete minimal surfaces[1]. This suggests that if such a theory was possible, then it would be very different from the theory developed in [F1, F2] (and also the Nevanlinna theory for holomorphic curves into $\mathbb{P}^n$). In fact, this is the direct motivation of our study. The following conjecture is not explicitly written in the literature but has been proposed many times as a problem :

Conjecture 1. The Gauss map of an algebraic minimal surface can omit at most two values.

We establish the value distribution theory of the Gauss map of algebraic minimal surfaces so that it is applicable to settle Conjecture 1. Therefore the value distribution theoretic property of the Weierstrass data of algebraic and pseudo-algebraic minimal surfaces is the main interest in this report. We study the Weierstrass data not as itself as objects defined on the basic domain, but as objects lifted to the universal covering surface $\mathbb{D}$, by using the methods of Nevanlinna theory coupled with the $\pi_1(M)$-action [2]. We have two reasons in doing so :

(1) We are interested in the general property of the Weierstrass data of (pseudo-) algebraic minimal surfaces. Especially, the upper bound of the totally ramified value number (TRVN) and the number of exceptional values of the Gauss map among the set of all (pseudo-)algebraic minimal surfraces.

(2) There is no canonical way of formulating Nevanlinna theory on the basic domain. If we lift the Weierstrass data on $\mathbb{D}$, then the problem is to study Nevanlinna theory of the lifted data under the presence of the action of $\pi_1(M)$ (M being the basic domain) on $\mathbb{D}$ (and moreover, to study the lifted version of the period condition in algebraic case).

Nevanlinna's Lemma on logarithmic derivative (the LLD, in short) is the most basic Diophantine-type inequality on which the whole Nevanlinna theory is built (see, for instance, [NO], [Ko1] and [Y]). Our attempt constructing Nevanlinna theory for the lifted Weierstrass data begins with establishing a version of the LLD applicable to the lifted Weierstrass data of (pseudo-)algebraic minimal surfaces. Rather surprisingly, the LLD applicable to the lifted Weierstrass data turns out to have nothing to do with the geometric origin of the Weierstrass data except that they are originally obtained by restricting to M meromorphic objects defined on $\overline{M}$. Indeed, we formulate our LLD in the folloing way : Let $g : \mathbb{D} \to \mathbb{P}^1$ be any meromorphic function on $\mathbb{D}$ whose height function $T_g(r)$

[1] Moreover, in the Nevanlinna theory for holomorphic curves into $\mathbb{P}^n$, the Nevanlinna defect is estimated by the integer $n + 1$

[2] If the universal covering surface is $\mathbb{C}$, then the theory reduces to the classical Nevanlinna theory. So, we always assume that the universal covering surface is always the disk $\mathbb{D}$.

has growth comparable to that of the Gauss map of some (pseudo-)algebraic minimal surface. Then we have

$$(m_{g,D}(r) - m_{g^{(1)},D^{(1)}}(r)) + m_{g^{(1)},S_\infty}(r) \leq (\kappa_g + \delta)\, T_g(r) /\!/_{E_\delta}$$

where D is any finite collection of disrtinct points of $\mathbb{P}^1$, κ_g is an invariant constructed from the meromorphic function $g : \mathbb{D} \to \mathbb{P}^1$ by

$$\kappa_g = \inf_\kappa \left\{ \kappa \;\middle|\; \int_0^1 \exp(\kappa_1\, T_g(r))dr = \infty \right\}$$

and finally $\delta > 0$ is taken to be arbitrarily small. The definition of κ_g implies

$$\kappa_g\, T_g(r) = \log \frac{1}{1-r} \quad \text{asymptotically} \quad \text{as} \quad r \to 1\ .$$

The above the LLD looks like a usual LLD (except that the right hand side is not of order $\log T_g(r)$ but of the form (const)$T_g(r)$) and only the meromorphic function g is involved. However, what we are interested in is the universal upper bound of κ_g for the lifted Gauss map g among all pseudo-algebraic minimal surfaces (this is interpreted as a Nevanlinna theoretic or "collective" Cohn-Vossen inequality). We would like to show

$$0 < \kappa_g \leq 2.72\ .$$

The outline of the proof I have in mind is outlined later. The number 2.72 looks close to $e = 2.718\cdots$. In fact, it becomes clear that any number which is slightly larger than e works. This is equivalent to proving the asymptotic estimate (as $r \to 1$)

$$T_g(r) \geq \frac{1}{2.72} \log \frac{1}{1-r}$$

for the lifted Gauss map of all pseudo-algebraic minimal surfaces. The inequality $\kappa_g \leq 2.72$ makes our LLD effective. We have an idea of the proof of the estimate $\kappa_g \leq 2.72$ for a pseudo-algebraic minimal surface M which consists of analyzing the effect of the Euclidean distortion of fundamental domains w.r.to the action of $\pi_1(M)$ on $\mathbb{D}$.

Although we establish an effective LLD applicable to the Gauss map of pseudo-algebraic minimal surfaces, it is still difficult to develop appropriate Nevanlinna theory for the Weierstrass data of algebraic minimal surfaces. The origin of the difficulty comes from the following two facts:

(1) In the pseudo-algebraic case, we must work on the Weierstrass data defined on the basic domain (a punctured Riemann surface M), or equivalently, on its universal covering $\mathbb{D}$ with the invariance under the $\pi_1(M)$-action.

(2) In the algebraic case, we must work under the stronger "invariance", namely, under the $\pi_1(M)$-invariance of the Weierstrass-Enneper representation (the "real" Abel-Jacobi integral). This invariance is called as the **period condition** in minimal surface theory. The period condition is a priori independent from the LLD. However, I have an idea that our effective LLD provides a Nevanlinna theoretic interpretation of the period condition of algebraic minimal surfaces, if it is applied to an appropriate pair of a holomorphic function on $\mathbb{D}$ and a divisor in $\mathbb{P}^1$ which encodes the period condition.

Therefore, our next objective is to analyze the period condition in the framework of Nevanlinna-Galois theory. In this direction, what we would like to prove is the following "Nevanlinna-Galois relation" :

$$2\,T_g(r) - J \le (3\kappa_g - 8)\,T_g(r)$$

for the Weierstrass data of algebraic minimal surfaces, where J is constructed from certain proximity functions involving g, dg and ω (we will come to this point later). The proof I have in mind is based on applying the effective LLD to the pair of a function constructed from g, dg and ω and a certain divisor which in combination *encode* the *period condition*. Once we prove the "Nevanlinna-Galois relation", we combine this with the basic geometric pattern from LLD to the Second Main Theorem to get the strong Second Main Theorem for the Gauss maps of algebraic minimal surfaces :

$$m_{g,D}(r) + N_{g,\mathrm{Ram}}(r) \le (\kappa_g + \delta)\,T_g(r) + (2\,T_g(r) - J)/\!\!/_{E_\delta} \le (2.88 + \delta)\,T_g(r)/\!\!/_{E_\delta}$$

where δ is any small positive number, D is any reduced divisor on $\mathbb{P}^1$ and 2.88 is $4(\kappa_g - 2)$ with $\kappa \le 2.72$. This will imply that for algebraic minimal surfaces

$$D_g \le \nu_g \le 2.88$$

holds.

Now I would like to proceed to the outline of the proof of Conjecture 1 I have in mind at this stage.

We start with $M = \overline{M}\backslash\{P_1, \ldots, P_n\}$ (a compact Riemann surface finitely many points removed) and the Weierstrass data (g, ω) on M satisfying the regularity condition on M. We may assume that the Gauss map g has no poles at $\{P_1, \ldots, P_n\}$. Then the completeness implies that ω has poles at $\{P_1, \ldots, P_n\}$. From "large scale" view point, our strategy is to follow the algebro-geometric argument in [KKM] (which generalizes that of Osserman [O1, O2]). In [KKM], we introduced the ratio $R := \frac{\int_M g^*\omega_{\mathrm{FS}}}{\int_M \omega_{\mathrm{hyp}}}$ of the Fubini-Study area aginst the hyperbolic area as a basic quantity to study, where the Fuibini-Study and the Poincaré metric has curvature 4π and -4π, repectively. The argument in [KKM] consists of two steps : (1) the Cohn-Vossen inequality $R \ge 1$ for pseudo-algebraic minimal surfaces (consequence from the Riemann-Roch and completeness of the algebraic minimal surface under consideration, where the completeness is used in the form that ω has poles at $\{P_1, \ldots, P_n\}$) and the Osserman inequality $R > 1$ for algebraic minimal surfaces. The period condition of algebraic minimal surfaces is used in the form that ω has poles at $\{P_1, \ldots, P_n\}$ of order ≥ 2 and the strict inequality $R > 1$ is a consequencde from this. (2) Then the Riemann-Hurwitz formula implies $\nu_g \le 2 + \frac{2}{R}$ (ν_g represents the totally ramified value number of $g : M \to \mathbb{P}^1$). This implies $\nu_g < 4$ for algebraic minimal surfaces. This is the outline of the argument of [KKM].

Now we proceed to introducing new ideas toward constructing the Nevanlinna-Galois theory, which I have in mind at this stage to settle Conjecture 1. First of all we introduce a Nevanlinna theoretic analogue of the basic quantity R in the following way. We define κ_g to be the infimum of numbers κ satisfying the property

$$\int_0^1 \exp(\kappa T_g(t))dt = \infty\ ,$$

where $T_g(r)$ is the Nevanlinna height function of g, i.e., the height transform of the Fubini-Study area function $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))$ $(0 < t < 1)$. If we define κ_{hyp} to be the infimum of numbers κ satisfying the property $\int_0^1 \exp(\kappa T_{\mathrm{hyp}}(t))dt = \infty$, where $T_{\mathrm{hyp}}(r)$ is the height transform of the hyperbolic area function $\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(t))$ $(0 < t < 1)$, we have $\kappa_{\mathrm{hyp}} = 2$. This observation supports that κ_g is a Nevanlinna theoretic analogue of the basic quantity R. Then we prove the "collective" version of the Cohn-Vossen inequality

$$\kappa_g \leq 2.72 \ ,$$

which is a Nevanlinna theoretic analogue of the Cohn-Vossen inequality $R \geq 1$. This becomes more conprehensible if we compare two asymptotic statements

$$\kappa_g \, T_g(r) = \log \frac{1}{1-r} \quad \text{and} \quad 2\, T_{\mathrm{hyp}}(r) = \log \frac{1}{1-r} \quad \text{as} \quad r \to 1 \ .$$

In fact, contrary to the classical Cohn-Vossen inequality, the collective Cohn-Vossen inequality has nothing to do with the 1-form part ω of the Weierstrass data (and therefore has nothing to do with the completeness of the (pseudo-)algebraic minimal surface under consideration) and holds for the lift, w.r.to the universal covering map $\pi : \mathbb{D} \to M$, of the restriction to M of any meromorphic function on $\overline{M}$ whose Nevanlinna height function has growth comparable to that of the Gauss map of some pseudo-algebraic minimal surface. The difference of the collective Cohn-Vossen inequality from the classical Cohn-Vossen inequality lies in the fact that we must consider the collection of fundamental domains cut out by the concentric discs $\{\mathbb{D}(t)\}_{0<t<1}$ (if we cut a fundamental domain by $\mathbb{D}(t)$ we loose neighborhoods of cusps) instead of a single whole fundamental dimain (which is the same as M). In the attempt of estimating the invariant κ_g, we must look at the ratio $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))/\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(t))$. Therefore what we should do is to extract useful information from the fact that the disk $\mathbb{D}(t)$ $(t < 1)$ in covered by the pieces of fundamental domains of various Euclidean shape. The difficulty is that although we can compare global integrals $\int_M g^*\omega_{\mathrm{FS}}$ and $\int_M \omega_{\mathrm{hyp}}$ by using the Riemann-Roch, we have no effective way of comparing $\int_{F\cap\mathbb{D}(t)} g^*\omega_{\mathrm{FS}}$ and $\int_{F\cap\mathbb{D}(t)} \omega_{\mathrm{hyp}}$ on the portion of the fundamental domain F cut out by the disk $\mathbb{D}(t) = \{|z| < t\}$. In order to compute the asymptotic behavior of the ratio $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))/\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(t))$ as $t \to 1$, we observe that the image of the circle $|z| = t$ ($t < 1$ is very close to 1) "concentrates" around the cusps $\{P_1, \ldots, P_n\}$. We call this concentration phenomenon as the "localization principle via iterate of parabolic translations". This localization principle implies that the set of "Euclidean distorted fundamental domains" dominates the set of all pieces of fundamental domains in $\mathbb{D}(t)$ (in number) as $t \to 1$. Therefore the localization principle allows us to compare $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))$ and $\mathrm{Area}_{\mathrm{hyp}}(D(t))$ only around a neighborhood of cusps. Indeed, we apply the idea of the *saddle point method* in the integration theory combined with the proof of the central limit theorem to effectively compare $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))$ and $\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(t))$. Here, in order to get an approximation of the ratio $\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(t))/\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(t))$, we apply the idea of the saddle point method to the collection of the ratios FS-area/hyp-area of various pieces of fundamental domains cut out by $\mathbb{D}(t)$. We then use the idea of the proof of the central limit theorem to examin the acuracy of the approximation. In conclusion, we have the collective Cohn-Vossen inequality $\kappa \leq 2.72$ as a Nevanlinna theoretic analogue of the classical Cohn-Vossen inequality $R \geq 1$.

As soon as we establish the collective Cohn-Vossen inequality $\kappa \le 2.72$, we can prove the "effective" Lemma on logarithmic derivative (the LLD) for meromorphic functions $f : \mathbb{D} \to \mathbb{P}^1$ whose height function has comparable growth as the lift, w.r.to the covering map $\pi : \mathbb{D} \to M$, of the restriction to M of the Gauss map of some pseudo-algebraic minimal surface : if D is any collection of distinct points of $\mathbb{P}^1$ we have

$$(m_{f,D}(r) - m_{f^{(1)},D^{(1)}}(r)) + m_{f^{(1)},S_\infty}(r) \le (2.72+\delta)\, T_f(r) \le (1+\delta)\, \log \frac{1}{1-r} /\!\!/_{E_\delta}$$

for any small $\delta > 0$, where $m_{f,D}(r)$ (resp. $m_{f^{(1)},D^{(1)}}(r)$) measures the approximation of f to D (resp. in the first jet level) and S_∞ is the infinity section in the projective completion of the tangent bundle $\mathcal{O}_{\mathbb{P}^1}(2)$.

Next issue to study is how to establish a Nevanlinna theoretic analogue of the strict inequality $R > 1$, which culminates in the translation of the period condition of algebraic minimal surfaces into a statement in Nevanlinna-Galois theory on $\mathbb{D}$ with $\pi_1(M)$)-action. The Weierstrass representation formula

$$\mathbb{D} \ni z \mapsto \Re\left(\int_{z_0}^{z} (1-g^2)\omega,\ \int_{z_0}^{z} i(1+g^2)\omega,\ 2\int_{z_0}^{z} g\omega\right) \in \mathbb{R}^3$$

gives a realization of the algebraic minimal surface in $\mathbb{R}^3$ and therefore the above integration changes by *pure imaginary* numbers under the action of $\pi_1(M)$ to $z \in \mathbb{D}$. From this ovservation it follows that if $H(z)$ represents the integral

$$H(z) = \int_{z_0}^{z} p(g)\omega$$

where $p(g)$ is a "generic" polynomial of deg ≤ 2 in g chosen from the $\mathbb{R}$-vector space

$$V := \mathbb{R}\,(1-g^2) + \mathbb{R}\,i(1+g^2) + \mathbb{R}\,(2g)\ ,$$

then the period condition is equivalent to

$$H(\gamma z) - H(z) \in i\,\mathbb{R}\ ,$$

i.e., $|e^H|$ is $\pi_1(M)$-invariant. In order to establish a Nevanlinna theoretic interpretation of the period condition, we collect points of $\mathbb{D}$ where the lifted Gauss map has poles. It follows from the $\pi_1(M)$-invariance of $|e^H|$ that the values of e^H at poles of g distribute along a finite number of circles $|w| = \alpha_i$ in $\mathbb{P}^1$ where w is an affine parameter of $\mathbb{P}^1$. Now the idea is to apply the effective LLD to the function e^H with the approximation target the values of e^H at poles of g. The choice of this approximation target is based on the observation that these values are the only information which is extracted from the effective LLD applied to e^H after the "averaging" procedure over various choice of $p(g)$ from V. Here, the "averaging" procedure is performed on $\mathbb{S}(V)$ w.r.to the invariant probability measure, where $\mathbb{S}(V)$ is the unit sphere in V w.r.to the metric having $\{1 - g^2, i(1+g^2), 2g\}$ as an ONB. To get a useful information from this procedure, we must look at the effect of the $\pi_1(M)$-action on the effective LLD applied to e^H. In our situation, the circle $|z| = r$ in $\mathbb{D}$ over which the integration is performed in the proximity function, has two interpretations. It is not only a circle in $\mathbb{D}$ but also its image in M makes sense. Indeed, the lifted Weierstrass data is defined on $\mathbb{D}(t)$ where we use the linear coordinate z to define derivatives, and the original Weierstrass data is defined on $\overline{M}$ where we use the local parameter ζ of $\overline{M}$ around $\{P_1, \ldots, P_n\}$ to define derivatives.

These two interpretations will give us two versions of the proximity functions (i.e., z- and ζ-proximity functions) and therefore two conclusions from the effective LLD applied to e^H. Moreover, at this stage, we essentially use the completeness of the algebraic minimal surface under consideration. This is translated into a statement that ω has poles at $\{P_1, \dots, P_n\}$, from which we have the Nevanlinna theoretic relation $m_{h^\sharp, S_\infty}(r) = 2\,T_g(r)$. Here $\omega = h(z)dz$ and $h^\sharp$ means the spherical image of $z \mapsto (g(z), h(z))$ in $\overline{T_{g(z)}\mathbb{P}^1}$, S_∞ being the infinity section of $\overline{T\mathbb{P}^1}$. The consequence from the z- and ζ-versions of the effective LLD's applied to e^H coupled with the completeness culminates in a single inequality ("Nevanlinna-Galois relation" for algebraic minimal surfaces)

$$2T_g(r) - J \le (3\kappa_g - 8)\,T_g(r)$$

valid for the Gauss map of any algebraic minimal surface, where J is a Nevanlinna theoretic function defined as

$$J = (m_{g^{(1)}, S_\infty}(r) - m^\zeta_{g^{(1)}, S_\infty}(r)) + (m_{h^\sharp, S_0}(r) - m^\zeta_{h^\sharp, S_0}(r))$$

constructed from the z- and ζ-proximity functions involving g, dg and ω (the function J arises naturally by applying the geometric pattern from the LLD to the SMT in Nevanlinna theory to the decomposition $dg = \frac{dg}{\omega} \cdot \omega$, see, for inctance, [Ko1]).

Finally, the Nevanlinna theoretic function

$$m_{g,D}(r) + N_{f,\mathrm{Ram}}(r)$$

is the Nevanlinna theoretic way of counting all multiple roots of the equation $g(z) \in D$, D containing exceptional values which are interpreted as multiple roots with multiplicity ∞. Considering $m_{g,D}(r) + N_{f,\mathrm{Ram}}(r)$ is an Nevanlinna theoretic analogue of the use of the Riemann-Hurwitz formula in the estimation of the totally ramified value number of $g : M \to \mathbb{P}^1$ in [KKM]. We apply the standard Nevanlinna calculus to the decomposition $g' = (g'/h)h$ (where $g'dz = dg$ and $h(z)dz = \omega$) to reduce the estimate (from above) of $m_{g,D}(r) + N_{f,\mathrm{Ram}}(r)$ to that of

$$\kappa_g\, T_g(r) + (2T_g(r) - J)\ ,$$

where κ_g is effectively estimated by the collective Cohn-Vossen inequality and $2T_g(r) - J$ is estimated by the "Nevanlinna-Galois relation".

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