FAMILIES OF CONFORMAL TORI OF REVOLUTION IN THE 3–SPHERE

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Abstract. For all positive integers \( n \) we construct a 1-parameter family of conformal tori of revolution in the 3-sphere with \( n \) bulges. These tori arise by Darboux transformations of constant mean curvature tori in the 3-sphere but do not have constant mean curvature in \( S^3 \).

1. Introduction

In a recent paper [2] it is shown that the multiplier spectral curve of a conformal torus \( f : T^2 \to S^4 \) is essentially given by the set of closed Darboux transforms of \( f \): to each multiplier on the spectral curve there exists a quaternionic holomorphic section with the given multiplier in the associated quaternionic holomorphic line bundle \( W \) of \( f \). The prolongation of the holomorphic section defines a new conformal torus \( \bar{f} \), and it turns out that \( f \) and \( \bar{f} \) satisfy a “weak enveloping” condition. Thus \((f, \bar{f})\) form a generalized Darboux pair. Classically, the Darboux transformation is defined for isothermic surfaces and a map \( f : M \to \mathbb{R}^3 \) is called a classical Darboux transform [3] of an isothermic \( f : M \to \mathbb{R}^3 \) if there exists a sphere congruence enveloping both \( f \) and \( \bar{f} \).

For every conformal torus the set \( H^0_h(W) \) of holomorphic sections with a given multiplier \( h \) is generically 1-dimensional, and at generic points the Darboux transformation preserves geometric properties: e.g., generic Darboux transforms of a constant mean curvature torus have constant mean curvature [4], and generic Darboux transforms of a Hamiltonian stationary torus are Hamiltonian stationary [5].

However, there exist examples, e.g. [7], of conformal tori which allow non-trivial multiplier on the spectral curve with high dimensional space of holomorphic sections. The existence of these singular multipliers should allow a deformation of the spectral curve; in the case of constant mean curvature tori in the 3-sphere of spectral genus zero [2] one can deform the spectral curve to obtain a family of Delaunay tori by removing this singularity of the spectral curve, and thus by adding geometric genus. By contrast the Darboux transformation preserves the geometric spectral genus [2] in the case when the Darboux transform is immersed but it may change geometric properties (e.g. break the constant mean curvature condition). In particular, the Darboux transformation at

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singular points is expected to allow to add or remove arithmetic genus of the spectral curve, and a thorough understanding of the singular points of the multiplier spectral curve may play an important role in understanding the reconstruction of conformal tori by their spectral data \cite{[3][1][1][9]} and the study of minimum energy tori in presence of a variational principle \cite{[11][17]}.

In this short note, we concentrate on the geometric properties of the Darboux transformation in the case when the conformal torus is a rectangular torus in the 3-sphere: in particular, we construct for each \( n \in \mathbb{N}, n \geq 2 \), a 1-parameter family of conformal tori of revolution in \( S^3 \) with \( n \) bulges which do not have constant mean curvature in \( S^3 \).

Using a similar argument we also construct for each \( a \in \mathbb{R}, a > 0 \), a 1-parameter family of cylinder of revolution with non-constant mean curvature.

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2. **Rectangular Tori**

In the following, we will apply the Darboux transformation on rectangular tori \( f : \mathbb{C}/\Gamma \to S^3 \subset \mathbb{R}^4 \) with lattice \( \Gamma = \mathbb{Z} + \frac{i}{2} \mathbb{Z}, u, v > 0 \), to construct the families of conformal tori of revolution. We identify Euclidean 4-space \( \mathbb{R}^4 = \mathbb{H} \) with the quaternions and parametrize a rectangular torus with parameters \( (u, v) \) by

\[
f(x, y) = \frac{uv}{\sqrt{u^2 + v^2}} \left( \frac{1}{u} e^{2\pi jux} + \frac{1}{v} e^{2\pi jvy} \right).
\]

In particular, we will use the fact that a rectangular torus \( f : T^2 \to S^3 \) is Hamiltonian stationary, and use the methods and settings developed in \cite{[3]} to compute the Darboux transforms of \( f \). We can write \( f \) as

\[
f(x, y) = e^{\beta} \left( \frac{1}{u} + i \frac{1}{v} \right) \left( \frac{-j}{2\pi} \right) g,
\]

where the so-called Lagrangian angle \( \beta \) is given by

\[
\beta(z) = 2\pi \langle \beta_0, z \rangle \quad \text{with} \quad \beta_0 = u - vi \in \Gamma^* = u\mathbb{Z} \oplus iv\mathbb{Z} ,
\]

and

\[
g = 2\pi \rho e^{\pi j(ux + vy)},
\]

with scale \( \rho = \frac{uv}{\sqrt{u^2 + v^2}} \in \mathbb{R} \). Here, we denote by \( \langle \cdot, \cdot \rangle \) the standard inner product on \( \mathbb{C} \). Moreover, the derivative of \( f \) can be written as

\[
df = 2\pi \rho e^{\pi j(ux + vy)} dz \cdot je^{\pi j(ux + vy)} = e^{\frac{i\rho}{2}} dz \cdot g,
\]

and \( f \) is a conformal immersion, that is \cite{[3]} Sec. 2.2]

\[
*df = N df = -df R.
\]

with left normal

\[
N = e^{\beta} i = e^{2\pi j(ux - vy)} i
\]

and right normal

\[
R = -g^{-1} \cdot i g = i e^{2\pi j(ux + vy)}.
\]
For every conformal immersion \( f : M \rightarrow \mathbb{H} \) with right and left normals \( R \) and \( N \) the normal bundle of \( f \) is given \([3]\) Sec 2.2] by
\[
\mathcal{N}_f = \{ x \in \mathbb{H} \mid N x R = -x \}.
\]
In particular, if \( f : M \rightarrow S^3 \subset \mathbb{H} \) is a conformal map into the 3-sphere, then \( n = N f = f R \) and \( f \) are unit normals. Thus, the second fundamental form \( \Pi_{33} \) of \( f \) as a map into \( \mathbb{H} \) computes with
\[
(X df(Y))^\perp = - < df(Y), dh(X) > n - < df(Y), df(X) > f
\]
as
\[
\Pi_{33} = \Pi_{33} - df^2 f.
\]
From this we see that the mean curvature vector \( \mathcal{H}_{33} \) in \( \mathbb{R}^4 = \mathbb{H} \) relates to the mean curvature vector \( \mathcal{H}_{33} = H_{33} n \) via
\[
\mathcal{H}_{33} = H_{33} n - f
\]
where \( H_{33} \) is the mean curvature of \( f \) in \( S^3 \). We denote by
\[
(dN)' = \frac{1}{2} (dN - N * dN) \quad \text{and} \quad (dN)' = \frac{1}{2} (dN + N * dN)
\]
the \((1,0)\) and \((0,1)\)-parts of the derivative of \( N \) with respect to the complex structure \( N \), and define \( H \) by \((dN)' = -df H\). Then it is shown in \([3] \) Sec. 7.2] that the mean curvature vector of a conformal immersion into \( \mathbb{R}^4 \) is given by
\[
\mathcal{H}_{33} = N \tilde{H}.
\]
Combining the previous equations, we see that the mean curvature \( H_{33} \) of a conformal immersion \( f : M \rightarrow S^3 \) of a Riemann surface \( M \) into \( S^3 \) is given by
\[
H_{33} = f H + N = \text{Re} (f H).
\]
In particular, since \( H \) computes in the case of Hamiltonian stationary Lagrangians \([3]\) to
\[
H = \pi g^{-1} \frac{1}{2} \frac{\partial}{\partial x} k
\]
the constant mean curvature of a rectangular torus in \( S^3 \) is given by
\[
H_{33} = \frac{1}{2} \left( \frac{u}{v} - \frac{v}{u} \right).
\]

3. THE DARBOUX TRANSFORMATION

We will briefly recall the construction of Darboux transforms in the case when \( f : M \rightarrow \mathbb{R}^4 \) is a conformal immersion from a Riemann surface into Euclidean 4-space. For the general case of conformal immersions into the 4-sphere and details of the construction compare \([2]\). In our situation, the associated quaternionic holomorphic line bundle of the immersion \( f \) can be identified with the trivial quaternionic bundle \( \mathbb{H} \) equipped with the (quaternionic) holomorphic structure \( D \) given by
\[
D \alpha := \frac{1}{2} (d \alpha + N * d \alpha),
\]
for $\alpha \in \Gamma(\mathbb{H})$ where $N$ is the left normal of $f : M \to \mathbb{R}^4 = \mathbb{H}$. We denote by $H^0(\mathbb{H}) = \ker D$ the set of holomorphic sections of the holomorphic line bundle $(\mathbb{H}, D)$. The *prolongation* of a local holomorphic section $\alpha \in H^0(\mathbb{H})$ is given by the local section

$$\psi = \begin{pmatrix} f^\nu + \alpha \\ \nu \end{pmatrix} \in \Gamma(\mathbb{H}^2)$$

of the trivial $\mathbb{H}^2$ bundle where $\nu$ is defined by $d\alpha = -df\nu$. Then $\psi$ spans locally a quaternionic line bundle $L$, and if $d\alpha$ is nowhere vanishing, the corresponding map $\hat{f} = f + \alpha\nu^{-1}$ is a branched conformal immersion into $\mathbb{R}^4$, a so-called a *Darboux transform* of $f$. If we denote by $T = f - \hat{f}$, then the derivative of $\hat{f}$ is given away from the zeros of $\alpha$ by

$$d\hat{f} = -T d\nu \alpha^{-1} T.$$ 

From $d\alpha = -df\nu$ we see that $df \wedge d\nu = 0$, in other words, $s d\nu = -R d\nu$ since $f$ has right normal $R$. In particular, $f$ has left normal

$$\hat{N} = -T R T^{-1}.$$ 

To compute the mean curvature vector $\hat{H}$ of the Darboux transform $\hat{f}$, it remains (2.1) to compute $\hat{H}$ using the defining equation $(d\hat{N})' = -d\hat{f} \hat{H}$. To this end, note that the derivative of $\hat{N}$ computes with $\hat{f} = f + T$ as

$$d\hat{N} = -* df T^{-1} + d\hat{f} T^{-1} \hat{N} - T dR T^{-1} + \hat{N} df T^{-1} - s d\hat{f} T^{-1}$$

so that the $(1,0)$-part of $d\hat{N}$ with respect to $\hat{N}$ is given by

$$(d\hat{N})' = d\hat{f} T^{-1} \hat{N} - T (dR)' T^{-1} - s d\hat{f} T^{-1}$$

where $(dR)' = \frac{1}{2}(dR + R * dR)$.

To obtain Darboux transforms which are globally defined, we consider holomorphic sections with multiplier, that is holomorphic sections of the trivial bundle $\mathbb{H}$ over the universal cover $M$ of $M$ which satisfy

$$\gamma^* \alpha = \alpha h_\gamma$$

with $h_\gamma \in \mathbb{C}$ for all $\gamma \in \pi_1(M)$. From $d\alpha = -df\nu$ and (3.1) we see that the prolongation $\psi$ of $\alpha$ has multiplier $h_\gamma$, that is $\gamma^* \psi = \psi h_\gamma$ for $\gamma \in \pi_1(M)$, so that $\psi$ defines a branched conformal immersion $\hat{f} : M \to \mathbb{H}$ if $\alpha$ is nowhere vanishing.

In the case when $f : T^2 \to S^4$ is a conformal 2-torus, the existence of global Darboux transforms is guaranteed by the link [2] between Darboux transforms and the *multiplier spectral curve* $\Sigma$ of $f$: to every multiplier $h \in \Sigma$ there exists at least one holomorphic section with multiplier $h$, and each such holomorphic section gives by prolongation a Darboux transform $\hat{f} : T^2 \to S^4$ of $f$. In other words, there is at least a Riemann surface worth of Darboux transforms of a conformal torus.

4. **Darboux Transforms of Hamiltonian Stationary Lagrangian Tori**

In the following we summarize notations and results of [8]. In the case of a Hamiltonian stationary Lagrangian torus $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ with Lagrangian angle $\beta = 2\pi \langle \hat{A}, \cdot \rangle$, every multiplier of a holomorphic section is of the form

$$h = h^{A,B} = e^{2\pi \langle \hat{A}, \cdot \rangle - i \langle \hat{B}, \cdot \rangle)}$$
with $A, B \in \mathbb{C}^2$ such that
\[
\Gamma_{A,B}^* = \{ \delta \in \Gamma^* \mid \delta \text{ satisfies } |\delta - B|^2 - |A|^2 = \frac{|\delta - B, A|^2}{4} \text{ and } \langle \delta - B, A \rangle = 0 \}
\]
is not empty. A holomorphic section $\alpha \in H^0(\mathbb{R}^2)$ with multiplier $h^{A,B}$ is called monochromatic if it is given by a Fourier monomial, that is if
\[
\alpha = \alpha_0 := e^{\frac{\pi i}{\beta_0}} (1 - k\lambda_0) e^{2\pi i \langle A, \cdot \rangle}
\]
is given by a single frequency $\delta \in \Gamma^*_{A,B}$ where $\lambda_0 := \frac{2}{\beta_0} (\delta - iA - B)$ and
\[
e_\gamma(z) := e^{2\pi i \langle \gamma, \cdot \rangle}.
\]
A polychromatic holomorphic section with multiplier $h^{A,B}$ is given by a non-trivial linear combination $\alpha = \sum_{\delta \in \Gamma^*_{A,B}} m_\delta \alpha_\delta$ of monochromatic holomorphic sections, $m_\delta \in \mathbb{C}$.

**Definition 1.** A branched conformal immersion $\hat{f} : M \to S^4$ is called a monochromatic Darboux transform (respectively polychromatic) if it is given by the prolongation of a monochromatic (respectively polychromatic) holomorphic section.

In $\mathbb{R}^4$, it is shown that all monochromatic Darboux transforms of a rectangular torus are after reparametrization again a rectangular torus. Moreover, polychromatic holomorphic sections with multiplier $h^{A,B}, A \neq 0$, only occur if $h^{A,B}$ is real. In particular, the corresponding Darboux transform coincides with a monochromatic Darboux transform.

To obtain new tori we therefore have to consider polychromatic Darboux transforms with $A = 0$:

**Theorem 1 ($\mathbb{R}^4$).** Let $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ be a Hamiltonian stationary torus in $\mathbb{R}^4$ with Lagrangian angle $\beta$ and $df = e^{\frac{\pi i}{\beta_0}} dz g$. Then every non-constant, polychromatic Darboux transform $\hat{f} : \mathbb{C}/\Gamma \to \mathbb{R}^4$ of $f$ with $A = 0$ is given by
\[
\hat{f} = f + e^{\frac{\pi i}{\beta_0}} \left( \sum_{t \in I_B} (1 + ke^{it}) m_t \bar{m}_t e^{2\pi i (t - e^{it})} (1 + ke^{it}) \sin t \right) \frac{1}{R\beta_0} g
\]
where the finite set
\[
I_B = \{ t \in [0, 2\pi] \mid B - \frac{\beta_0}{2} e^{it} \notin \Gamma^*_{0,B} \} \neq \{0, \pi\}
\]
parametrizes the admissible frequencies and $m_t \in \mathbb{C}$ are chosen so that the map
\[
R = \left| \sum_{t \in I_B} m_t \sin t e^{2\pi i (t - e^{it})} \right|^2 + \left| \sum_{t \in I_B} m_t e^{2\pi i \sin t} e^{2\pi i (t - e^{it})} \right|^2
\]
is nowhere vanishing.

5. **Families of conformal tori of revolution in $S^3$**

In this section we discuss the Darboux transforms of a rectangular torus $f : T^2 \to S^3$. If the parameters $(u, v)$ of $f$ satisfy $u \geq \sqrt{3}v$, we show that there exist polychromatic Darboux transforms which are conformal tori in the 3-sphere.
Fix \( n \in \mathbb{N}, n \geq 2 \), and consider all rectangular tori \( f \) in the 3-sphere with parameters \((u, v)\) satisfying \( u^2 + v^2 (1 - n^2) \geq 0 \). The multiplier \( h = h^{0, B} \) with
\[
B = \frac{\beta_0}{2} + \frac{nvi}{2} - \sqrt{\frac{u^2 + v^2 (1 - n^2)}{2}}.
\]
has \( \dim H^0_{0, B} = \left| \Gamma^* \right| \geq 2 \) since
\[
\delta_+ = \frac{\beta_0}{2}, \quad \delta_- = \frac{\beta_0}{2} + nvi \in \Gamma^* \cap \{ \delta \in \Gamma^* \mid \left| \delta - B \right|^2 = \left| \frac{\beta_0}{4} \right| \},
\]
which shows that \( f \) allows polychromatic Darboux transforms. In particular, we see that
\[
-\lambda^\pm = \frac{2}{\beta_0} (B - \delta^\pm) = c^\pm + is^\pm
\]
with
\[
c^\pm = \frac{\mp nvi - u\delta}{r^2}, \quad s^\pm = \frac{\pm uv - sv}{r^2}
\]
(5.1) and
\[
s := \sqrt{r^2 - u^2 v^2}
\]
so that (4.1) all monochromatic holomorphic sections with multiplier \( h^{0, B} \) are given by
\[
\alpha^\pm := \alpha \delta^\pm = e^{i(uw - vy)} (1 + js^\pm + kc^\pm) e^{i(ux + nvy)}.
\]
Using Theorem 1, we see that the corresponding polychromatic Darboux transforms are given by
\[
\hat{f} = e^{i\theta} \left( -\left( \frac{j}{u} + \frac{k}{v} \right) + \left( \sum_{i,j \in \{\pm\}} a_{ij} \frac{u - iv}{Rr^2} \right) \frac{a}{2\pi} \right)
\]
where
\[
a_{ij} = (1 + ke^{it_j})m_i \bar{m}_j e^{2\pi i \langle \delta^\pm, \delta^\pm \rangle} (1 + ke^{it_j}) \sin t_j,
\]
and \( m^\pm \in \mathbb{C} \) have to be chosen so that
\[
2R(y) = \left| m^+_+ s^+_+ + m^-_- s^-_e^{i\theta} \right|^2 + \left| m^+_+ s^+_+ + m^-_- s^-_e^{-i\theta} \right|^2
\]
is nowhere vanishing where
\[
\tilde{y} := 2\pi nvy.
\]
Using (5.1) and \( |z + w|^2 = |z|^2 + |w|^2 + 2\Re (z\bar{w}) \) the denominator \( R \) simplifies to
\[
R = \left| m^+_+ s^+_+ + m^-_- s^-_e^{i\theta} - \frac{2n^2 |1 - n^2|}{r^4} \Re (m^+_+ \bar{m}_-(v^2 - n^2 v^2 - ivn) e^{-i\theta}) \right.
\]
In general, the above Darboux transforms will be conformal immersions into the 4-sphere. However, there exist constants \( m^+_+, m^-_- \in \mathbb{C} \) so that the polychromatic Darboux transform is a conformal immersion into the 3-sphere.

**Theorem 2.** For each \( n \in \mathbb{N}, n \geq 2 \), there exists a 1-parameter family of conformal tori of revolution in the 3-sphere with \( n \) bulges. Only one conformal torus in this family has constant mean curvature in \( S^3 \).
Proof. We consider the case that \( m \pm = m \). In this case, \((5.1)\)
\[
s_\pm^2 + s_\pm^2 = \frac{2n^2}{r^4} \left( u^2 (1 + n^2) + v^2 (1 - n^2) \right)
\]
shows that
\[
\frac{Rr^4}{2|m|^2 \rho^2} = u^2 (1 + n^2) + v^2 (1 - n^2) + (1 - n^2) \left( (r^2 - n^2 v^2) \cos(\hat{y}) - \sin \sin(\hat{y}) \right).
\]
From \((5.2)\)
\[
a_{ij} = |m| \left( (1 + ke^{it_1})e^{2\pi i(\delta_j - \delta_i)} (1 + ke^{it_j}) \sin t_j \right)
\]
we see that \( \hat{f} \) is independent of the choice of \( m \in \mathbb{C} \). In particular, we may assume from now on that \( m = 1 \). Then
\[
a_{\pm, \pm} = 2ke^{it} s_{\pm} = 2(ke^{it} s_{\pm} + j s_{\pm}^2)
\]
is independent of \( z = x + iy \) while
\[
a_{\pm, \mp} = (1 + ke^{it} e^{\mp \hat{y}} + (ke^{it} - e^{it} e^{-it}) s_{\mp} e^{\pm \hat{y}}
\]
is independent of \( x \). In particular, the Darboux transform is given by \( \hat{f} = f + T \) where
\[
T = e^{\frac{it}{\rho}} r g, \quad \tau = \tau_0 + i \tau_1 \text{ with }
\]
\[
\tau_0(y) = \frac{j \tan^2}{\pi R(y)} \in j \mathbb{R}
\]
and
\[
\tau_1(y) = \frac{n(su \cos \hat{y} + s^2 \sin \hat{y}) + j(s^2 + (s^2 \cos \hat{y} - \sin \sin \hat{y}))}{\pi v R(y)} \in \mathcal{E} = \text{Span}\{1, j\}
\]
and
\[
R(y) = u^2 (1 + n^2) + v^2 (1 - n^2) + (1 - n^2)(s^2 \cos \hat{y} - \sin \sin \hat{y})
\]
Writing \( \hat{f} = e^{\frac{it}{\rho}} r \hat{g} \) where
\[
\hat{\tau} = \tau + \sigma, \quad \sigma = -\frac{1}{2\pi} \left( \frac{j}{u} + \frac{k}{v} \right),
\]
a lengthy, but straightforward, computation shows that
\[
|\hat{\tau}|^2 = \frac{1}{4\pi^2 \rho^2}
\]
In other words, \( \hat{f} : T^2 \to S^3 \) is a conformal map into the 3–sphere. On the other hand, we have
\[
\hat{f} = e^{i\pi (ux - vy)} (\tau_0 - \frac{j}{2\pi u} + i(\tau_1 - \frac{j}{2\pi v}))2\pi \rho \rho j e^{i\pi (ux + vy)} = e^{2j\pi ux} \kappa_0 + i e^{2j\pi vy} \kappa_1
\]
where \( \kappa_0 = (2\pi \tau_0 j + \frac{1}{n}) \rho \) is a real valued function, and both \( \kappa_0 \) and \( \kappa_1 = (2\pi \tau_1 j + \frac{1}{v}) \rho \) only depend on \( y \), that is, \( \hat{f} \) is a surface of revolution in \( S^3 \). Note that for \( u = v \sqrt{n^2 - 1} \), the Darboux transform \( \hat{f} \) is a rectangular torus with constant mean curvature in the 3–sphere. If \( u \neq v \sqrt{n^2 - 1} \) then \( \kappa_0 \) is extremal at
\[
y_k = \frac{1}{2nv} \left( \frac{1}{\pi} \arctan \left( \frac{vn}{\sqrt{u^2 + v^2 (1 - n^2)}} \right) + k \right), \quad k = 0, \ldots, 2n - 1,
\]
in particular, \( \hat{f} \) is a torus of revolution in \( S^3 \) with \( n \) bulges. If \( \hat{f} \) had constant mean
Figure 1. Darboux transforms of rectangular tori with parameter (1.8,1) and (2.1, 1)

Figure 2. Non-embedded Darboux transform of the rectangular torus with parameter (2.6, 1)

curvature in $S^3$ and is not a rectangular torus, we would $\hat{f}$ expect to be a Delaunay torus and thus to be parametrized by elliptic functions. However, the Darboux transformation essentially preserves spectral genus [2], and in our case $\hat{f}$ is parametrized by (rational functions of) trigonometric functions. Indeed, we compute the mean curvature $H_{s3} = \text{Re}(\hat{f}^\dagger\hat{H})$ of $\hat{f}$ in $S^3$ explicitly when $u \neq v\sqrt{n^2 - 1}$: first, $\hat{H}$ is defined by $(d\hat{N})' = -d\hat{f}\hat{H}$, that is [33]

$$\hat{H} = -T^{-1}\hat{N} + \hat{f}_x^{-1}(Tr_x + \hat{N} \hat{f}_x)T^{-1}$$

where $(dR)'' = r_x dx + r_y dy$ with

$$r_x = \pi g^{-1} j(u + v)$$

Moreover, we compute $\hat{f}_x = e^{i\Phi}2\pi j u \hat{\nu}_0 g = e^{i\Phi} q g$ with real valued

$$q = 2\pi j u \hat{\nu}_0 = 1 - \frac{2u^2 n^2}{R}$$

The left normal of $\hat{f}$ is given by [32]

$$\hat{N} = e^{i\Phi} \tau i \tau^{-1} e^{-i\Phi}$$
so that \( \hat{f}_t = e^{it\hat{\gamma}}e^{-it\hat{\tau}} \) with \( \hat{\gamma} = \hat{\tau} \gamma \tau^{-1} \) and
\[
\gamma = -i + q^{-1} \pi \tau j(u\bar{v} - v) + \tau i \tau^{-1}.
\]
Next observe that for any \( z, w \in \mathbb{H} \) we have \( \Re (zw^{-1}) = \Re w \) which shows that
\[
\Re (\gamma) = q^{-1} \pi \Im \left( v\tau_0 + u\tau_1 \right),
\]
and the mean curvature of \( \hat{f} \) in \( S^3 \) is, using \( \hat{\tau} = \tau + \sigma \), given by (2.2)
\[
\hat{H}_{S^3} = \Re \left( \sigma \gamma \tau^{-1} \right) + \frac{\pi}{q} \Im \left( v\tau_0 + u\tau_1 \right),
\]
(5.5)
Furthermore, for \( \lambda = \lambda_0 + i\lambda_1 \in \mathbb{H}, \lambda_0, \lambda_1 \in \mathbb{C} \), we have with \( \hat{\tau} = -\tau \)
\[
\Re (\sigma \lambda \tau^{-1}) = -\frac{1}{2\pi |\tau|^2} \Im \left( \tau_0 \left( \frac{\lambda_0}{u} + \frac{\lambda_1}{v} \right) + \tau_1 \left( \frac{\bar{\lambda}_0}{v} - \frac{\bar{\lambda}_1}{u} \right) \right)
\]
and we get for both \( \lambda = -i \) and \( \lambda = \tau i \tau^{-1} \)
\[
\Re (\sigma \lambda \tau^{-1}) = \frac{1}{2\pi |\tau|^2} \Im \left( \frac{\tau_0}{v} - \frac{\tau_1}{u} \right).
\]
whereas for \( \lambda = q^{-1} \pi \tau j(u\bar{v} - v) \)
\[
\Re (\sigma \lambda \tau^{-1}) = \frac{1}{2q} \left( \frac{v}{u} + \frac{u}{v} \right) - \frac{1}{uvq |\tau|^2} \left( v\Im \tau_0 + u\Im \tau_1 \right)^2.
\]
A straightforward computation shows that
\[
\pi uv |\tau|^2 = \Im \left( v\tau_0 + u\tau_1 \right)
\]
so that (5.5) simplifies to
\[
\hat{H}_{S^3} = \frac{1}{\pi |\tau|^2} \Im \left( \frac{\tau_0}{v} - \frac{\tau_1}{u} \right) + \frac{1}{2q} \left( \frac{v}{u} + \frac{u}{v} \right).
\]
(5.6)
For \( y = 0 \) one easily obtains with
\[
\hat{R} := 2\hat{u}^2 + \hat{v}^2 \left( 1 - n^2 \right) \left( 2 - n^2 \right) \quad \text{and} \quad \hat{q} := 2\hat{u}^2 - n^2 \hat{v}^2
\]
that
\[
\tau_0(0) = j \frac{n^2 u}{\pi R}, \quad \text{and} \quad \tau_1(0) = \frac{1}{\pi v R} \left( vn^2 s + i(q - n^2 v^2) \right)
\]
so that (5.6) becomes
\[
\hat{H}_{S^3}(0) = \frac{r^2 \left( vn^2 u - \bar{v} \right) - 2\hat{q}(n^2 - 1)v^2}{2(n^2 - 1)uv\hat{q}}.
\]
Similarly, for \( y = \frac{1}{2n} \) we obtain with
\[
\hat{R} := 2\hat{u}^2 + \hat{v}^2 \left( 1 - n^2 \right)
\]
that
\[
\tau_0 = j \frac{u}{\pi R}, \quad \text{and} \quad \tau_1 = -\frac{s}{\pi R}
\]
which gives
\[
\hat{H}_{S^3}(\frac{1}{2n}) = \frac{2\hat{u}^2 v^2 \left( n^2 - 1 \right) - r^2 \hat{R}}{2(n^2 - 1)uv^3}.
\]
In particular, if \( \tilde{f} \) has constant mean curvature in \( S^3 \) then \( \tilde{H}_{S^3}(0) = \tilde{H}_{S^3}(\frac{1}{2\eta}) \) gives
\[
 r^2 (v^2 n^4 - \tilde{q})v^2 - 2\tilde{q}(n^2 - 1)v^4 = 2u^2 v^2 (n^2 - 1)\tilde{q} - r^2 \tilde{R}\tilde{q}
\]
which is equivalent to
\[
 4r^2 (u^2 + v^2 (1 - n^2))^2 = 0,
\]
that is, \( u = v \sqrt{n^2 - 1} \). Finally, we notice that all rectangular tori are, up to reparametrization \( \tilde{z} = vz, z = x + iy \), rectangular tori with parameter \( (\frac{u}{v}, 1) \). Thus, we obtain for each \( n \in \mathbb{N}, n \neq 1 \), a 1-parameter family of tori of revolution in \( S^3 \) with \( n \) bulges, each torus given by a polychromatic Darboux transform of a rectangular torus with parameter \( (u, 1), u \geq \sqrt{n^2 - 1} \).

6. Polychromatic Darboux Transforms of Cylinders

We use similar methods to compute polychromatic Darboux transforms of a standard cylinder \( f : M \to \mathbb{R}^3 \). To stay close to the notations and computations in the previous sections, our maps \( f \) will take values in \( \text{Span}\{1, j, k\} \). Note that in this case \( f \) has mean curvature \( B \) given by
\[
 H_{S^3} = -Hi
\]
where \( H \) is again given by \( (dN)' = -dfH \). A standard cylinder
\[
 f(x, y) = \frac{1}{u} e^{2\pi jux} + 2\pi ky
\]
is then a Hamiltonian stationary immersion with harmonic left and right normals
\[ N = e^{2\pi jux}i \quad \text{and} \quad R = ie^{2\pi jux} \]
and Lagrangian angle \( \beta(z) = 2\pi(\beta_0, z) \) with \( \beta_0 = u \). Moreover, we have \( df = \frac{\sqrt{2}}{4} dz \) with \( g = 2\pi je^{\pi jux} \), and \( f = e^{\frac{\sqrt{2}}{4}}(-\frac{1}{2} + \pi i y) \). With the same methods as before (with the obvious adaptations to the situation of a cylinder), we consider for \( a \in \mathbb{R} \) all cylinder with \( u \geq a \), and obtain again for
\[ B = \frac{1}{2}(u + ai - \sqrt{u^2 - a^2}) \]
holomorphic sections with multiplier. The corresponding frequencies are
\[ \delta_+ = \frac{u}{2}, \quad \delta_- = \frac{u}{2} + ai \in \Gamma_{0,1} = \{ \delta \in u\mathbb{Z} + i\mathbb{R} + \frac{\beta_0}{2} | |\delta - B| = \frac{1}{2}\} \]
and the monochromatic holomorphic sections with multiplier \( h^{0,1} \) are
\[ \alpha_{\pm} = \frac{1}{u}e^{\pi jux}(u \pm ja - k\sqrt{u^2 - a^2}) \pi i (\sqrt{u^2 - a^2} + ay) \].
Again, we apply Theorem \( \Box \) with constants \( m_+ = m_- = 1 \), and obtain, after a similar computation as in the case of rectangular tori, the monochromatic Darboux transforms of a cylinder for \( h^{0,1} \) as
\[ \tilde{f} = e^{\frac{\sqrt{2}}{4}}(j\tau_0 + i\tau_1) \frac{g}{\pi} = 2(-e^{2\pi jux}\tau_0 + k\tau_1) \]

**Figure 4.** Darboux transforms of rectangular tori with parameter (4.3,1), (5.3,1) and (6.3,1)
with real valued functions
\[
\tau_0(x, y) = \frac{1}{u} \left( -\frac{1}{2} + \frac{1}{R} \right)
\]
and
\[
\tau_1(x, y) = \pi y + \frac{1}{a R} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y} \left( \frac{a}{u^2} \sqrt{u^2 - a^2} \right) \right).
\]
Here, we have with \( \tilde{y} = 2\pi n y \)
\[
\tilde{R}(y) = \frac{u^2}{4a^2} R = 1 - \left( 1 - \frac{a^2}{u^2} \right) \cos \tilde{y} + \frac{a}{u^2} \sqrt{u^2 - a^2} \sin \tilde{y}.
\]
and thus, both \( \tau_0 \) and \( \tau_1 \) only depend on \( y \). In particular, \( \tilde{f} \) is a surface of revolution in the 3-space spanned by 1, \( j, k \), and obviously, \( f \) is a round cylinder whenever \( u = a \) for \( a \in \mathbb{R} \).

We now compute the mean curvature of \( \tilde{f} \). To that end, we observe that \( \tilde{f} = f + T \) with
\[
T = \frac{2}{R} \left( -\epsilon e^{2\pi j nx} \frac{\kappa_0}{a} \frac{k}{u} + \frac{1}{a} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y} \left( \frac{a}{u^2} \sqrt{u^2 - a^2} \right) \right) \right),
\]
and the left normal \( \tilde{N} \) of \( \tilde{f} \) is given \[\text{(5.2)}\] by
\[
\tilde{N} = -\frac{i e^{2\pi j nx} (\kappa_0^2 - \kappa_1^2) + 2j \kappa_0 \kappa_1}{\kappa_0^2 + \kappa_1^2}.
\]
with
\[
\kappa_0 = -\frac{1}{a} \quad \text{and} \quad \kappa_1 = \frac{1}{a} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y} \left( \frac{a}{u^2} \sqrt{u^2 - a^2} \right) \right)
\]
real valued. As before, we compute with \[\text{(5.5)}\]
\[
\tilde{H} = \frac{i (\tilde{R} \kappa_0) + \frac{1}{a R_0}}{\lambda},
\]
and, by evaluating at \( y = 0 \) and \( y = \frac{1}{2\pi} \), we see that \( \tilde{H} \) constant is equivalent to \( a = u \).

We summarize

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Polychromatic Darboux transforms of cylinder of radius 2.1 and 2.9}
\end{figure}
Theorem 3. For all $a \in \mathbb{R}$, $a > 0$, the Darboux transformation gives a 1-parameter family of cylinders of revolution which are not constant mean curvature cylinders in the 3-space.

REFERENCES


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