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LOOP GROUPS AND SURFACES WITH SYMMETRIES

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1. Introduction

For more than ten years now the loop group method has been employed sucessfully in the construction of "integrable surfaces" in $\mathbb{R}^3, S^3$ and $\mathbb{H}^3$ (see e.g. [3] and references there). Much of the effort went into finding the right set-up and in proving basic theorems, in particular with regard to how to incorporate into the loop group method symmetries of the surfaces to be constructed.

For surfaces in 3-dimensional space forms two seemingly quite different approaches exist: one for surfaces for which conformal coordinates are natural and another one for surfaces for which asymptotic line coordinates are natural. It turns out that all these theories can be complexified [16] and that all known integrable surface classes can be obtained as real forms of these complex constant mean curvature surfaces [19]. In this note we will present old and new results about symmetries for surface classes, where conformal coordinates are most natural. In particular we will show how one can construct CMC-surfaces with symmetries in $\mathbb{H}^3$ with mean curvature $0 \leq H < 1$.

Symmetries for integrable surface classes where asymptotic line coordinates are most natural have been treated for the first time in [12] and [13]. And one-parameter groups of intrinsic isometries are under investigation [6] and [13]. A few comments about this can be found in the concluding remarks at the end of this note. In the long run all these symmetries for the different types of surface classes should become part of a theory of complex CMC surfaces.

After the introduction (section 1) we recall the basic loop group scheme and present the basic facts about the incorporation of (orientation preserving) symmetries into this scheme. Given an immersion $\phi : \mathbb{D} \to \mathbb{R}^3 \cong su(2)$ in conformal coordinates, a symmetry of ϕ is a pair(γ, R), where γ is a biholomorphic automorphism of $\mathbb{D}$ and R a rigid motion of $\mathbb{R}^3 \cong su(2)$ satisfying the symmetry relation

$$(S) \ : \ \phi(\gamma.z) = R.\phi(z).$$

Section 2 discusses what (S) means for the moving coordinate frame of ϕ, the coordinate frames of the associated family ϕ_λ of ϕ and the "holomorphic extended frame" and the "potentials" associated with the loop group method under discussion.

In section 3 the case is discussed, where the group of symmetries is the fundamental group of some Riemann surface. More precisely, let $\Gamma \subset \mathbb{D}$ denote the group of deck

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transformations of some Riemann surface M. We discuss in this section, what properties the potential of some associated family of CMC-immersion needs to have in order to generate a CMC- immersion that descends to M for some value λ_0. As a result, one obtains that all such (associated families of) CMC-immersions can be obtained from potentials which are invariant under Γ.

In section 4, the case of rotational symmetries of finite order is discussed. (The case of one-parameter groups of symmetries has been discussed by B.Smyth and leads to members of the associated family of Delaunay surfaces. In this context also see Theorem 3.3 of DoCarmo and Dajczer, reproven by G.Haak.) The discussion of this section is new. It (largely) justifies to use potentials satisfying $\gamma^*\eta = A\eta A^{-1}$ for the construction of CMC-immersions with finite order rotational symmetry A.

In section 5 we discuss CMC-surfaces in $\mathbb{H}^3$. Primarily we present the loop group approach to CMC-surfaces in $\mathbb{H}^3$ with mean curvature $0 \leq H < 1$, which has not been discussed in the literature so far. (But see [1].) It is very parallel to the case of CMC-surfaces in $\mathbb{R}^3$. Proofs and technical details are contained in [11].

Section 6 consists of concluding remarks about symmetries of other surface classes and also about Smyth type surfaces.

2. Symmetries for CMC-surfaces in $\mathbb{R}^3$

2.1. **The basic set-up for CMC-surfaces in $\mathbb{R}^3$.** We recall briefly the basic set-up. For details we refer to [3],[4],[5].

Let $\phi : \mathbb{D} \to \mathbb{R}^3 \cong su(2)$ be a CMC-immersion, where $\mathbb{D}$ is some simply connected open subset of $\mathbb{C}$. We will always assume that ϕ is conformal. With ϕ one associates a moving frame $F \in SU(2)$. We will use the same notation for the members of the associated family of ϕ. The parameter for the associated family will be $\lambda \in S^1$. We would like to note, however, that in most situations λ can be chosen arbitrarily in $\mathbb{C}^*$. The loop group method consists of the following two "strands":

Backward Strand

(B1) Let $\phi : \mathbb{D} \to \mathbb{R}^3 \cong su(2)$ be a CMC-immersion.

(B2) Let F denote the moving coordinate frame of ϕ. Introduce the loop parameter λ into the Maurer-Cartan form $F^{-1}dF$ of F and integrate to obtain the "extended frame" (again denoted by) F.

(B3) Find a matrix function $V_+ : \mathbb{D} \to \Lambda^+ SL(2,\mathbb{C})_\sigma$ which has a holomorphic extension in λ from S^1 to $\|\lambda\| < 1$ such that $C(z,\lambda) = F(z,\bar{z},\lambda)V_+(z,\bar{z},\lambda)$ is holomorphic for $z \in \mathbb{D}$.

(B4) Compute the "holomorphic potential" associated with C, i.e., compute the Maurer-Cartan form $\eta = C^{-1}dC$ of C.

Forward Strand

(F1) Consider a holomorphic potential $\eta = \sum_{-1}^{\infty} \eta_j \lambda^j$

(F2) Solve the ODE $\eta = C^{-1}dC$ on $\mathbb{D}$.

(F3) Perform the Iwasawa decomposition $C = FV_+$

(F4) Compute the CMC-immersion ϕ from F.

Then F will be the extended frame of ϕ.

We would also like to point out that the computation of ϕ can be achieved by differentiation of F with respect to λ, i.e. by an application of the Sym-Bobenko formula.

As was shown in [4], the forward strand always produces CMC-immersions and all CMC-immersions can be obtained this way.

2.2. **Symmetries for CMC-immersions in $\mathbb{R}^3$.** A simple way of thinking about a symmetry of an immersion ϕ is to consider a rigid motion R, $Rp = Up + a$, where U is linear and $a, x \in \mathbb{R}^3 = su(2)$, such that $R\phi(\mathbb{D}) = \phi(\mathbb{D})$. Under mild assumptions, like if the induced metric is complete, there exists some conformal map $\gamma : \mathbb{D} \to \mathbb{D}$ such that

$$(S) \;\; \phi(\gamma.z) = R.\phi(z)$$

for all $z \in \mathbb{D}$

holds. Therefore, more rigorously, a *symmetry* of ϕ will always denote a pair (γ, R) satisfying (S). For simplicity of presentation we will always assume that R is orientation preserving. Thus γ actually is a holomorphic automorphism of $\mathbb{D}$.

The quantities discussed in the previous section all have a specific transformation behaviour with respect to a given symmetry.

On the frame level, it is well known that a linear, orientation preserving rigid motion is of the form $Ax = UxU^{-1}$ with $U \in SU(2)$ and $a \in su(2)$.

Thus, given (γ, R) as above we obtain

Backward Strand with Symmetries

(BS1) $\phi(\gamma.z) = U\phi(z)U^{-1} + a$ for all $z \in \mathbb{D}$.

From this it is easy to derive for the coordinate frame

(BS2a) $F(\gamma.z, \overline{\gamma.z}) = UF(z, \bar{z})k(z, \bar{z})$ for all $z \in \mathbb{D}$ and some diagonal matrix k.

Introducing λ as usual leads to

(BS2b) $F(\gamma.z, \overline{\gamma.z}, \lambda) = \chi(\lambda)F(z, \bar{z}, \lambda)k(z, \bar{z})$ for all $z \in \mathbb{D}$.

Note that $k(z, \bar{z})$ does not acquire a λ-dependence. Moreover, $\chi(\lambda = 1) = U$ and $\chi(\lambda) \in SU(2)$ for all $\lambda \in S^1$.

The behaviour of $C(z, \lambda)$ under the induced symmetry transformation is

(BS3) $C(\gamma.z, \lambda) = \chi(\lambda)C(z, \lambda)W_+(z, \lambda)$ for all $z \in \mathbb{D}$.

Note, here $W_+ : \mathbb{D} \to \Lambda^+ SL(2, \mathbb{C})_\sigma$ has a holomorphic extension in λ from S^1 to $\|\lambda\| < 1$.

Finally, for the potential one obtains the transformation behaviour

(BS4) $\gamma^*\eta = W_+^{-1}\eta W_+ + W_+^{-1}dW_+$.

Hence the pullback of η under γ is a gauge transform of η.

For the forward strand this means to start with some γ and some W_+.

Forward Strand with Symmetries

(FS1) Consider a holomorphic potential $\eta = \sum_{-1}^{\infty} \eta_j \lambda^j$ satisfying

$\gamma^*\eta = W_+^{-1}\eta W_+ + W_+^{-1}dW_+$ for some $W_+ : \mathbb{D} \to \Lambda^+ SL(2, \mathbb{C})_\sigma$ which has a holomorphic extension in λ from S^1 to $\|\lambda\| < 1$.

(FS2) Solving the ODE $\eta = C^{-1}dC$ on $\mathbb{D}$ one obtains $C(\gamma.z, \lambda) = \rho(\lambda)C(z, \lambda)W_+(z, \lambda)$ for all $z \in \mathbb{D}$.

Note, ρ depends on the intial condition of C. Thus, for an arbitrary initial condition, $\rho(\lambda)$ will not be unitary for all $\lambda \in S^1$. However, from the backward strand we know that this monodromy matrix needs to be unitary in order to be associated with a symmetry of the immersion derived from η.

(FS3) Perform the Iwasawa decomposition $C = FV_+$. If for some initial condition of C, $\rho(\lambda) \in SU(2)$ for all $\lambda \in S^1$, then F acquires the transformation behaviour $\gamma^* F = \rho F k$, where k is independent of λ.

(FS4) Compute the CMC-immersion ϕ from F (e.g. by applying the Sym-Bobenko formula). Then $\gamma^*\phi = \rho\phi\rho{-}1 + \frac{d}{dt}\rho \cdot \rho^{-1}$, where $\lambda = e^{it}$.

Justifications and explanations can be found in [8], [9].and [14]

3. Incorporating the Fundamental group

If one wants to apply the results discussed in the previous section, then one needs to start from some potential satisfying $(FS1)$ for which the monodromy matrices ρ are unitary for all $\lambda \in S^1$.

It turns out that the gauge property of $(FS1)$ can be replaced by some easy to handle condition in the cases where γ runs through the fundamental group of some surface, or if γ has finite order.

Theorem 3.1. *([10]) Let $\phi : M \to su(2)$ be a CMC immersion, where M is a non-compact Riemann surface. Then there exists some potential η which is invariant under $\pi_1(M)$ and which induces, for $\lambda = 1$, the given immersion ϕ up to some translation.*

Theorem 3.2. *Assume η is a potential which is invariant under $\pi_1(M)$, where M is some non-compact Riemann surface. Assume that the monodromy matrices of C corresponding to $\gamma \in \pi_1(M)$ are all unitary for $\lambda \in S^1$. Then, for $\lambda = 1$, the immersion associated with C descends to an immersion from M into $su(2)$ if and only if for all $\gamma \in \pi_1(M)$*

$\rho(\lambda = I, \gamma) = \pm I$ for all $\gamma \in \pi_1(M)$,

$\frac{d}{d\lambda}\rho(\lambda,\gamma)_{|\lambda=1} = 0.$

Thus, if we start with an invariant potential, for which C has unitary monodromy matrices relative to $\Gamma = \pi_1(M)$, then Γ acts as a group of symmetries on the immersion derived from C.

If, in addition, the two closing conditions of the theorem above are satisfied, then the immersion descends to M.

In this spirit, Haak [18] gave a simple proof for a result, originally due to DoCarmo and Dajczer:

Theorem 3.3. *A CMC-surface is helicoidal if and only if it is contained in the associated family of a Delaunay surface.*

Another example of such an application is

Theorem 3.4 ([15])**.** *Assume η is a potential defined on some open strip $\mathbb{S}$ containing the real line and assume that η is invariant under the translation $z \to z+1$, and satisfies -$\overline{\eta(\bar z, \lambda)}^t = \eta(z,\lambda)$, then C with initial condition $C(0,\lambda) = I$ has unitary monodromy and* *the surfaces associated with C have a symmetry of the form* $\phi(z+1,\lambda) = U\phi(z,\lambda)U^{-1} + a$*. Moreover, if the monodromy matrix satisfies for* $\lambda = 1$ *the closing conditions, then* $\phi(z,\lambda = 1)$ *descends to a CMC-immersion of the cylinder* $\mathbb{S}/\mathbb{Z}$*.*

The discussion of CMC cylinders was started in the dissertation of Martin Kilian (University of Massachusetts, 2000). The first CMC-cylinders with umbilical points were constructed in [10]. A coarse classification of CMC-cylinders has been carried out in [15]. A final classification of all CMC-cylinders is still open.

4. Symmetries of finite order

Consider a CMC-immersion $\phi : \mathbb{D} \to su(2)$ and assume that it admits a symmetry (R,γ), $Rp = ApA^{-1} + a$, of finite order n. Then R has finite order n and we can assume that R is linear of order n. In particular, we will assume from now on $a = 0$. Moreover, we have

$$\phi(\gamma^n z) = \phi(z) \ \text{ for all } \ z \in \mathbb{D}. \tag{1}$$

It is natural now to consider the group

$$\Gamma_\phi = \{\delta : \mathbb{D} \to \mathbb{D} \ \text{ biholomorphic}, \ \ \phi(\delta z) = \phi(z) \ \text{ for all } \ z \in \mathbb{D}\}. \tag{2}$$

Clearly, $\gamma^n \in \Gamma_\phi$. It is known by Proposition 2.8 of [8] that Γ_ϕ acts freely and discontinously so that ϕ actually descends to a CMC-immersion from the Riemann surface $M = \Gamma_\phi \backslash \mathbb{D}$ to $su(2)$. Let χ denote the respresentation of Γ_ϕ induced by $(BS2b)$.

From $(BS2a)$ we know

$$F(\gamma.z, \overline{\gamma.z}) = AF(z,\bar{z})k(z,\bar{z},\gamma) \ \text{ for all } \ z \in \mathbb{D} \tag{3}$$

and after introducing λ we obtain from $(BS2b)$ the formula

$$F(\gamma.z, \overline{\gamma.z}, \lambda) = \rho(\lambda)F(z,\bar{z},\lambda)k(z,\bar{z},\gamma) \ \text{ for all } \ z \in \mathbb{D}. \tag{4}$$

Clearly, the first formula yields

$$F(\gamma^n.z, \overline{\gamma^n.z}) = A^n F(z,\bar{z})\hat{k}(z,\bar{z};\gamma) \ \text{ for all } \ z \in \mathbb{D}. \tag{5}$$

where $A^n = \pm I$ and $\hat{k}$ is some diagonal matrix independent of λ.

As a consequence, the last formula yields

$$F(\gamma^n.z, \overline{\gamma^n.z}, \lambda) = \rho(\lambda)^n F(z,\bar{z},\lambda)\hat{k}(z,\bar{z};\gamma) \ \text{ for all } \ z \in \mathbb{D}. \tag{6}$$

Since $\gamma^n \in \Gamma_\phi$, we also have

$$F(\gamma^n.z, \overline{\gamma^n.z}, \lambda) = \chi(\lambda,\gamma^n)F(z,\bar{z},\lambda)h(z,\bar{z}) \tag{7}$$

for some diagonal matrix h.

The general case considered above has not been investigated yet. However, in many examples it happens that γ has a fixed point z_*. In this case, we assume at $z = z_*$, rotating ϕ if necessary, without loss of generality $F = I$ and $C = I$ for every λ. We note that this changes the matrix A accordingly.

Under these assumptions, the last four formulas yield

- $A = k(z_*, \bar{z}_*, \gamma)^{-1}$. In particular, A is a diagonal matrix and independent of λ.
- $\rho(\lambda) = A$ is independent of λ.
- $A^n = \hat{k}(z_*, \bar{z}_*, \gamma)^{-1} = \pm I$.
- $\chi(\lambda = 1; \gamma^n) = \pm I$

We summarize this in

Theorem 4.1. *Consider a CMC-immersion $\phi : \mathbb{D} \to su(2)$ and assume that it admits a symmetry (R, γ), $Rp = ApA^{-1} + a$, of finite order n. Assume moreover, that γ has a fixed point $z_* \in \mathbb{D}$.*

(1) *If $F(z_*, \bar{z}_*, \lambda) = I$, then A is a $\lambda-$independent diagonal matrix of finite order and we have*

$$F(\gamma.z, \overline{\gamma.z}, \lambda) = AF(z, \bar{z}, \lambda)k(z, \bar{z}, \gamma) \ \text{ for all } \ z \in \mathbb{D}. \tag{8}$$

(2) *Conversely, if A is diagonal, then there exists some matrix $B(\lambda)$ such that $B(\lambda)F(z_*, \bar{z_*}, \lambda) = I$ and (8) holds for $\hat{F}(z, \bar{z}, \lambda) = B(\lambda)F(z, \bar{z}, \lambda)$.*

This situation can be obtained under slightly more general assumptions:

Theorem 4.2. *Consider a CMC-immersion $\phi : \mathbb{D} \to su(2)$ and assume that it admits a symmetry (R, γ), $Rp = ApA^{-1} + a$, of finite order n. If $\gamma^n = id$, then γ has a fixed point in $\mathbb{D}$ and after normalizing F or A we obtain the assumptions of the last theorem: Hence we can assume without loss of generality*

$$F(\gamma.z, \overline{\gamma.z}, \lambda) = AF(z, \bar{z}, \lambda)k(z, \bar{z}, \gamma) \ \text{ for all } \ z \in \mathbb{D}. \tag{9}$$

Moreover, if $\Gamma_\phi = \{I\}$, then $\gamma^n = id$.

Since a symmetry of finite order is frequently associated with some γ which has a fixed point, the results just stated imply that the monodromy matrix of a symmetry of finite order can in many cases be assumed to be constant. Also note that in these cases, A can be assumed to be a diagonal matrix, $A = diag(\delta, \delta^{-1})$, whose diagonal entries are $2n-$th roots of unity. Such symmetries can be obtained from potentials which also have a simple transformation behaviour.

Theorem 4.3. *Consider a CMC-immersion $\phi : \mathbb{D} \to su(2)$ and assume that it admits a symmetry (R, γ), $Rp = ApA^{-1} + a$, of finite order n and that γ has the fixed point $z_* = 0$. Then ϕ can be obtained over $\mathbb{D}^* = \mathbb{D} \setminus 0$ from a potential satisfying*

$$\gamma^* \eta = A \eta A^{-1} \tag{10}$$

Proof. Following $(B3)$ we choose any matrix function $V_+ : \mathbb{D} \to \Lambda^+ SL(2, \mathbb{C})_\sigma$ which has a holomorphic extension in λ from S^1 to $\|\lambda\| < 1$ such that

$C(z, \lambda) = F(z, \bar{z}, \lambda)V_+(z, \bar{z}, \lambda)$ is holomorphic for $z \in \mathbb{D}$. It is straightforward to verify that $C(\gamma.z, \lambda) = AC(z, \lambda)W_+(z, \lambda, \gamma)$ and that W_+ satisfies the cocycle condition of group cohomology. We want to apply the proof of [17], Theorem 28.4. For this we need to consider a Riemann surface X and its universal cover Y. We set $X = \mathbb{D}^*/G$, where G is the group generated by γ and Y as the universal cover of $\mathbb{D}^*$. In view of [2] we can conclude as in [17], Theorem 28.4, that there exists a matrix function $h_+(z,\lambda) \in \Lambda^+ SL(2,\mathbb{C})_\sigma$ such that $W_+(z,\lambda,\gamma) = h_+(\gamma.z,\lambda)h_+(z,\lambda)^{-1}$. Hence $\hat{C}(z,\lambda) = C(z,\lambda)h_+(z,\lambda)$ satisfies $\hat{C}(\gamma.z,\lambda) = AC(z,\lambda)$. Note that γ acts on Y as $z \to z + \frac{2\pi i}{n}$ and the generator of the fundamental group of C^* is $z \to z + 2\pi i$. Clearly, $A = \exp(\frac{2\pi i}{2n}\sigma_3)$. Therefore $D(z) = exp(-\frac{2\pi i}{n} z\sigma_3)$ satisfies $D(\gamma.z) = D(z)A^{-1}$. As a consequence, the Maurer-Cartan form of $\tilde{C} = \hat{C}D(z)$ satisfies $\gamma^*\eta = A\eta A^{-1}$. Since $A^n = \pm I$ it follows that η descends to $\mathbb{D}^*$, proving the claim. □

Remark 4.4. *We would expect that applying a Birkhoff decomposition along a circle in the $z-$domain one will be able to gauge the given potential η into another one, producing the smae surface, which, however, satisfies (10) on all of $\mathbb{D}$.*

The last result gives a clear indication of what potentials will yield surfaces admitting some rotational symmetry associated with a γ having a fixed point.

Theorem 4.5. *Let γ denote the rotation about 0 of order n in $\mathbb{D}$ and assume η is a potential defined on $\mathbb{D}$ satisfying (10). Then η induces a CMC-immersion from $\mathbb{D}$ into $su(2)$ with rotational symmetry A of order n about the $z-$axis.*

Proof. The transformation formula (10) implies that the solution to $dC = C\eta$, $C(0,\lambda) = I$ for all λ, implies $C(\gamma.z,\lambda) = AC(z,\lambda)A^{-1}$. Then, following $(FS3)$ we consider the Iwasawa decomposition $C = FV_+$, where we can assume that the diagonal coefficients of V_+ at λ^0 are positive, thus making the decomposition unique. In particular, this implies that F and V_+ are equal to I at z_*. Then $AFA^{-1}AV_+A^{-1} = ACA^{-1} = \gamma^*C = \gamma^*F\gamma^*V_+$ implies $AFA^{-1} = \gamma^*F$ and $AV_+A^{-1} = V_+$. Since A does not depend on λ, it follows that the CMC-immersion ϕ obtained from F via the Sym-Bobenko formula satisfies $\phi(\gamma.z,\lambda) = A\phi(z,\lambda)A^{-1}$. □

Remark 4.6. *Since A is diagonal and independent of λ, condition (10) applies to all the coefficients of η in a power series expansion in λ. Thus there are very many CMC-immersions with rotational symmetry. A simple example is given by the Smyth surfaces, see section 4 of* [9].

5. CMC-surfaces in $\mathbb{H}^3$ of mean curvature $0 \le H < 1$

We will work with the Minkowski space model of $\mathbb{H}^3$. Thus we work in the real four-dimensional vector space Herm(2,$\mathbb{C}$) of two by two complex hermitian matrices. The Minkowski inner product is given by $\langle X, Y\rangle = -$ trace $XE_2Y^tE_2$, where E_2 is the off-diagonal hermitian matrix with $(2,1)-$component i. Then $\mathbb{H}^3$ is the subset of Herm(2,$\mathbb{C}$) defined by the conditions $\det A = 1$ and trace $A > 0$. The group SL(2,$\mathbb{C}$) acts transitively on $\mathbb{H}^3$ by $X \to gXg^\star$.

Consider a conformal CMC-immersion $f : \mathbb{D} \to \mathbb{H}^3$ [11]. Let e^u denote the conformal factor of the metric induced by f. Thinking for the moment of $\mathbb{R}^4$ with Minkowski inner product one forms the moving frame $\Psi = (f, n, e^u/2f_y, e^u/2f_x)$, where n denotes the normal of f and the subscripts denote differentiation. This matrix is in the Lorentz group. Since $\mathrm{Sl}(2,\mathbb{C})$ is a finte cover of this group we lift Ψ. This lift will be denoted by $\hat{\Phi}$. To $\hat{\Phi}$ we apply the usual loop group procedure and insert a loop parameter λ. This way one obtains an extended frame, again denoted by Φ. It has the Maurer-Cartan form

$$\hat{\Phi}^{-1}\hat{\Phi}_z = \begin{pmatrix} u_z/4 & \frac{\lambda^{-1}}{2}(H+1)e^{u/2} \\ -\lambda^{-1}Qe^{-u/2} & -u_z/4 \end{pmatrix} \tag{11}$$

$$\hat{\Phi}^{-1}\hat{\Phi}_{\bar{z}} = \begin{pmatrix} -u_{\bar{z}}/4 & \lambda\bar{Q}e^{-u/2} \\ -\frac{\lambda}{2}(H-1)e^{u/2} & u_{\bar{z}}/4 \end{pmatrix}. \tag{12}$$

The Sym-type formula is $f = \hat{\Phi}\hat{\Phi}^{\star}$.

In contrast to the case of CMC-immersions into $\mathbb{R}^3$ one needs to distinguish in $\mathbb{H}^3$ three cases, namely $0 \leq H < 1$, $H = 1$ and $H > 1$. The remaining possibilities can be subsumed by some scaling. The case $H > 1$ is equivalent to CMC-immersions in $\mathbb{R}^3$. The case $H = 1$ has been studied extensively. We will therefore concentrate on the case $0 \leq H < 1$. We set $H = \tanh(q)$ with $q \geq 0$ and introduce the gauged lift $\Phi = \hat{\Phi} diag(e^{(q+\pi i)/4}, e^{-(q+\pi i)/4})$. Clearly, in terms of Φ the Sym-type formula now is $f = \Phi\delta\Phi^{\star}$, with $\delta = diag(e^{-(q+\pi i)/2}, e^{(q+\pi i)/2)})$. Using the additional abbreviations $\mathcal{H} = ie^{-q}(H+1)$, $\nu = e^{-q/2}\lambda$ and $\mathcal{Q} = -iQ$ we obtain $\mathcal{H} = ie^{-q}(H+1) = -ie^{q/2}(H-1)$ and the Maurer-Cartan form of ϕ takes the form

$$\Phi^{-1}\Phi_z = \begin{pmatrix} u_z/4 & -\frac{1}{2}\nu^{-1}\mathcal{H}e^{u/2} \\ \nu^{-1}\mathcal{Q}e^{-u/2} & -u_z/4 \end{pmatrix} \tag{13}$$

$$\Phi^{-1}\Phi_{\bar{z}} = \begin{pmatrix} -u_{\bar{z}}/4 & -\nu\bar{\mathcal{Q}}e^{-u/2} \\ \frac{1}{2}\nu\mathcal{H}e^{u/2} & u_{\bar{z}}/4 \end{pmatrix}. \tag{14}$$

This form of the Maurer-Cartan form is exactly as in $\mathbb{R}^3$, but the entries have quite a different meaning. However, one feature stays preserved: The Maurer- Cartan form is fixed under some involution. While in $\mathbb{R}^3$ this is the involution defining the unitary group inside Sl(2,$\mathbb{C}$), here it is the loop group involution

$$\tau : g(\nu) \to \mathcal{R}(g(i/\bar{\nu})^{\star}\mathcal{R}^{-1}, \tag{15}$$

where $\mathcal{R}$ denotes the diagonal matrix with with $(2,2)-$entry $\sqrt{i}$. It is important to note that the involution τ cannot be derived from a finite dimensional setting. Therefore, in contrast to all other integrable surface classes, the case under consideration here is not the "loopification" of a finite dimensional setting. In slightly different notation this involution was already used in [1]. This involution also shows up naturally in Kobayashi's classification [19] of all integrable surface classes which are associated with SL(2,$\mathbb{C}$).

The setting just described naturally permits a loop group procedure as in [4]. In particular, one can find holomorphic extended frames by solving a $\bar{\partial}-$problem or meromorphic extended frames via Birkhoff splitting. The Maurer-Cartan forms of such extended frames serve as potentials. As a matter of fact, these potentials are any twisted loop matrices in sl(2,$\mathbb{C}$) and their loop parameter ν has only the powers $-1, 0, 1, 2, ...$ As a matter of fact, the potentials of the $\mathbb{R}^3$ case and the $\mathbb{H}^3$ case are identical. There are two important differences, however: the Iwasawa splitting for the $\mathbb{H}^3$ case and the $\mathbb{R}^3$ case are relative to different groups, and in the end one needs to restrict ν to the circle of radius $e^{-q/2}$ for the $\mathbb{H}^3$ case and to the unit circle for the $\mathbb{R}^3$ case.

For details we refer to [11].

Concerning symmetries one can follow the pattern of the $\mathbb{R}^3$ case verbatim. However, due to the fact that the Iwasawa splitting in the $\mathbb{H}^3$ case is not global, many new features arise. More precisely, there are two open and disjoint Iwasawa cells and the frames become very singular on the boundary between these two cells. The surfaces, however, sometimes stay nonsingular, sometimes are singular. Moreover, even where on can build into the construction of the surface potential features that imply symmetries or local ends, one will need to expect additional features due to the "cell-crossing".

Here is a simple example:

Surface of Revolution

Consider the potential

$$\eta = Adz = \begin{pmatrix} 0 & \nu^{-1}a + \nu b \\ \nu^{-1}b - \nu a & 0 \end{pmatrix} dz, \tag{16}$$

where $a, b \in \mathbb{R}$ and $b^2 - a^2 + ab(e^q - e^{-q} = 1/4$. Next we consider the ODE $dC = C\eta$. It has the solution $C = \exp(Az)$, where we have chosen the intial condition I at $z = 0$. It is easy to check that $\tau(A) = -A$ holds. Therefore, for every $p \in i\mathbb{R}$ we obtain $\gamma^\star C = C(p, \nu)C$, where $\gamma(z) = z + p$ and $C(p, \nu)$ is fixed by τ. As a consequence, if $C = \Phi V_+$ denotes the (unique) Iwasawa decomposition of C, then $C(p, \nu)\Phi V_+ = C(p, \nu)C = \gamma^\star C = \gamma^\star \Phi \gamma^\star V_+$ has the Iwasawa decomposition $\gamma^\star \Phi = C(p, \nu)\Phi$ and $\gamma^\star V_+ = V_+$. The next to last equation can be interpreted as an action of $C(p, \nu)$ on Φ. It induces the symmetry of the immersion $f = \Phi \delta \Phi^\star$:

$$\gamma^\star f = C(p, \nu) f \mathbb{C}(p, \nu)^\star. \tag{17}$$

Specializing ν to $e^{-q/2}$ we obtain a one-parameter group of symmetries of the immersion $f_0(z) = f(z, e^{-q/2})$ given by $R(t) = C(it, e^{-q/2})$. It is easy to verify that $R(t)$ has the eigenvalues $\exp(\pm it/2)$. Therefore $f_0(z + it) = f_0(z)$ for $t = 2\pi ki$ and f_0 is a surface of revolution.

The following picture (produced by Shimpei Kobayashi) shows the surface of revolution associated with H = tanh (0.3), a=1, and b = -1.46328. The sphere represents the unit (Poincare) ball. One sees that the surface passes through the boundary of $\mathbb{H}^3$ without singularities. The frame, however, is very singular. The part of the surface inside the ball is obtained from the potential and the function $C(z, \nu)$ by Iwasawa splitting $C = \Phi V_+$. The part outside of the ball is obtained from C by the splitting $C = \Phi \omega W_+$, where ω is the off-diagonal matrix in Sl(2,$\mathbb{C}$) with $(1, 2)-$entry λ. This corresponds to the fact that C is in the open Iwasawa cell containing I in the first case and in the second open Iwasawa cell in the second case. For more details and explanations we refer to [11].

6. Concluding remarks

(1) As mentioned already above, the discussion presented so far basically applies to all integrable surfaces, like the ones listed in [19], which can be investigated best in conformal coordinates. If the isotropy group of the associated symmetric space is compact, the analogy is very close, the treatment basically identical. If the isotropy group of the associated symmetric space is non-compact, then the analogies will be very strong, but there will be singularities due to the non-global nature of the loop group splittings involved.

(2) Symmetries for the cases in [19], which can be investigated best in asymptotic line coordinates, have not been investigated so far. A first step in this direction is contained in [12] and [13]. In some sense also this case follows a pattern quite analogous to the one

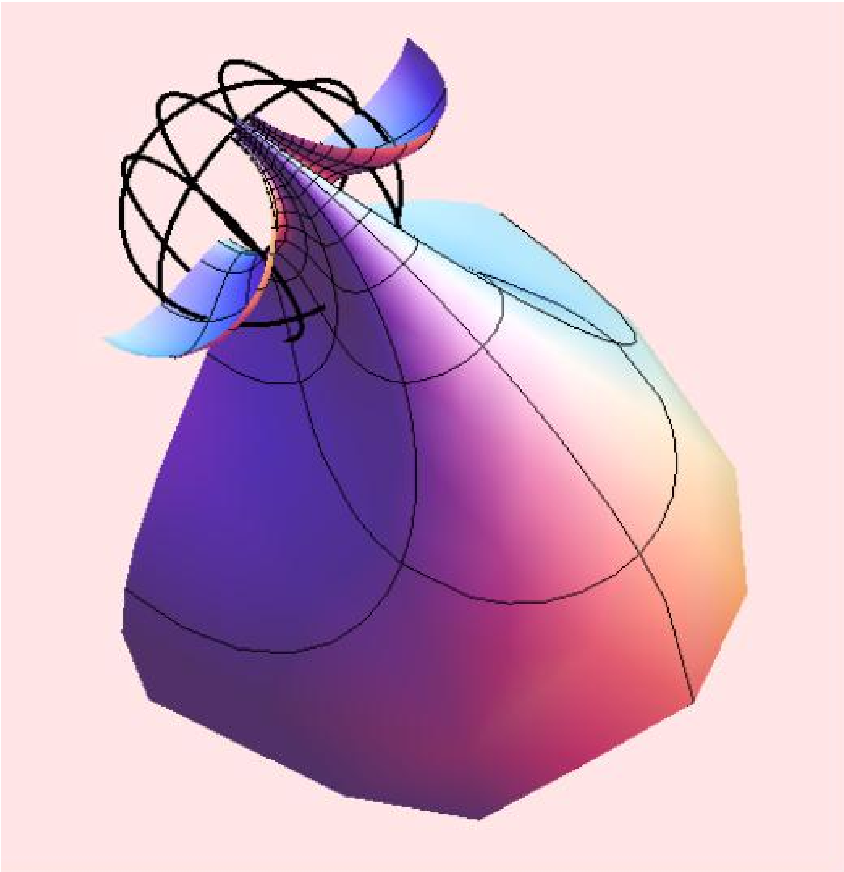

FIGURE 1. surface of revolution

presented in this note. On the other hand, the two independent variables x and y, as opposed to z and $\bar{z}$, require a technically different procedure. It is hoped that, eventually, there will be a theory of symmetries of the complex CMC-surfaces which give rise to all real integrable surfaces.

(3) More general than symmetries as discussed so far are (intrinsic) isometries of the induced metric. In particular, one-parameter groups of self-isometries relative to the induced metric fall into this category. For the case of CMC-surfaces in $\mathbb{R}^3$ with complete induced metric, these are the Smyth surfaces. In the case of more general integrable surfaces, where conformal coordinates are best suited for their description, it is not clear complete surfaces are the right subclass to be considered or what type of surface class should be singled out. A first paper in this direction [7] considers the general case for

spacelike CMC-surfaces in Minkowski space $\mathbb{R}^{2,1}$. In view of [6] one is interested in surfaces, for which the Iwasawa splitting is global.

(4) It is not clear at this point, whether intrinsic (non-extrinsic) one-parameter groups of self-isometries relative to the induced metric are of interest for those integrable surfaces, for which asymptotic line coordinates are best suited for their description.

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