Natural Classifications of Knots

Alessandro FLAMMINI and Andrzej STASIAK

ABSTRACT

The principal objective of the knot theory is to provide a simple way of classifying and ordering all the knot types. Here, we propose a natural classification of knots based on their intrinsic position in the knot space that is defined by the set of knots to which a given knot can be converted by individual intersegmental passages. In addition, we characterize various knots using a set of simple quantum numbers that can be determined upon inspection of minimal crossing diagram of a knot. These numbers include: crossing number; average three-dimensional writhe; number of topological domains; and the average relaxation value.

Keywords: knots; topology; knot invariants

1. Introduction

During the 1860s, British physicist William Thomson (now remembered as Lord Kelvin) proposed that different types of atoms were simply different types of knots made of vortices of ether (a hypothetical substance believed to be necessary for the propagation of electromagnetic radiation; Thomson 1869). As we now know, further development of physics negated the existence of ether and provided another explanation of distinct properties of various atoms. However, the hypothesis of Kelvin needs to be seen as a proposal that real elementary particles (which Thomson thought were atoms) can have a form of knots. This latter interpretation has recently found a spectacular confirmation (Buiny & Kephart 2003; Buiny & Kephart 2005). A class of elementary particles known as glueballs have such a spectrum of mass that clearly indicates that these particles exist as knotted quantum chromodynamics flux lines (Ralston 2003), where the flux lines follow paths of ideal knots, i.e. the shortest possible cylindrical tubes that can be closed into a given knot type (Moffatt 1990; Katritch et al. 1996; Faddeev & Niemi 1997). Inspired by this recent discovery, we present a natural classification of knots (with up to 9 crossings) that reflects the relationships between various knots and that takes into account earlier observations that the writhe, which is a measure of chirality of closed curves in a three-dimensional
space, is additive and quantized in the case of ideal geometric representations of knots (Katritch \textit{et al.} 1997; Pieranski 1998; Cerf & Stasiak 2000; Devlin 2001; Pieranski & Przybyl 2001; Cerf & Stasiak 2003).

2. Methods

To obtain a random configuration of a given knot type, we start with a polygonal configuration that resembles the axial trajectory of the so-called ideal knots of a given type (Katritch \textit{et al.} 1996). These starting configurations are then evolved as non-phantom chains by applying random crankshaft moves (Vologodskii \textit{et al.} 1992). The non-phantom evolution is achieved by accepting only these crankshaft moves, during which there are no segment–segment passages. After 20 000 of such accepted random moves, when the configuration is sufficiently randomized, the polygon is further evolved by random crankshaft rotations, but, in addition, moves that result in just one intersegmental passage are also accepted. The occurrence of intersegmental passages was detected by the checking for intersections between the surface of revolution generated by the rotating portion of the chain and segments of the subchain that was not rotated. After the first move that resulted in just one intersegmental passage, the evolution is terminated and the knot type of the polygonal chain is determined by the calculation of its HOMFLY polynomial (Freyd \textit{et al.} 1985; Ewing & Millett 1996; Dobay \textit{et al.} 2003). The entire procedure is repeated 20 000 times for each knot type with up to 9 crossings in the minimal crossing representation.

3. Results

\textbf{(a) The evolutionary tree of knots}

We classify knot types according to their reciprocal kinship in the knot space. For knot types that are directly related to each other, we consider the knot types that can be converted into each other by one intersegmental passage (figure 1c). Using a biological example, we consider a pair of knot types as related to each other when a DNA knot of a given type can be converted into a knot of another type by one catalytic cycle of a DNA topoisomerase (Darcy & Sumners 2000; Flammini \textit{et al.} 2004). We build a genealogical, or rather an evolutionary, tree of knots by investigating the inter-knot transitions resulting from random intersegmental passages, occurring in random configurations of each of the 101 types of knots that have up to 9 crossings in their standard diagrams with minimal numbers of crossings. The detailed procedure is described in \S 2. In brief, 20 000 random configurations of a given knot constructed out of 32 segment-long equilateral polygons are permitted to undergo random crankshaft motion, and after the first intersegmental passage, the evolution is terminated and the knot type of the polygonal chain is characterized by the calculation of its HOMFLY polynomial (Freyd \textit{et al.} 1985; Ewing & Millett 1996; Dobay \textit{et al.} 2003). For the construction of the tree of knots, we take into account only one-passage connectivity between the knots that differ in their minimal number of crossings, where, in each pair of such knots, the simpler one (with smaller minimal number of crossings) is considered as a predecessor and the more complex one as a
Natural classification of knots

Figure 1. Knots and their crossings. (a) Signs of crossings. Once the orientation of the closed curve in space defining a knot is consistently traced (as in (b)), it is easy to determine the sign of all perceived crossings in a knot. In a positive crossing, the overlying direction vector can be aligned with the underlying one by an anticlockwise rotation that is smaller than 180°. A corresponding clockwise rotation of the overlying direction vector would be required to align it with the underlying one in a negative crossing. (b) Characters of crossings. Double-arc regions (bigons) in oriented diagrams of knots reveal whether the orientation vectors on the opposing segments run in parallel or antiparallel directions. Crossings ‘belonging’ to parallel double-arc fields (P) have parallel character, while those belonging to antiparallel double-arc fields (A) have antiparallel character. The Alexander-Briggs notation of knots like 5_2, for example, uses two numbers, where the main number indicates the minimal number of crossings a given knot can have in its projection and the index number indicates the tabular position of this knot among the knots with the same minimal number of crossings (Rolfsen 1976; Adams 1994). Index letters ‘R’ and ‘L’ indicate the right- and left-handed enantiomorph of a given knot type. (c) A pair of knots that can be converted into each other by one intersegmental crossing. The region of the passage is marked in grey and the more complex knot (successor) gained two right-handed parallel crossings as compared with the predecessor knot. Therefore, the writhe of the successor knot (6_{2R}) has increased by 20/7, as compared with achiral knot 4_1 that has the writhe 0.

successor (figure 1c). Most often successors have two more crossings than predecessors have. Therefore, we concentrate first on the predecessors and the successors that differ in their minimal number of crossing by 2. A given knot type frequently has more than one predecessor with two fewer crossings. In such a case, we measure the relative order of kinship between the successors and the predecessors by determining the probability of relaxation of individual knot types towards their respective predecessors. We include this information in the constructed tree of knots (figure 2). In some knot types, random intersegmental passages lead with nearly identical probability (relative differences smaller than 5%) to two or more different predecessors. In such a case, we assign the same degree of kinship between such a successor knot and each of its respective
predecessors. When a given knot type does not have a predecessor with two fewer crossings, we look for its predecessors with a maximal possible number of crossings and indicate this accordingly on the evolutionary tree of knots.

It was observed earlier that the three-dimensional writhe, which is a measure of chirality of closed curves in space, is additive and quantized for axial trajectories of ideal geometric representations of knots and that this applies also to the mean writhe of statistical ensembles of random knots of a given type (Katritch et al. 1997; Pieranski 1998; Cerf & Stasiak 2000; Pieranski & Przybyl 2001). The additivity of writhe implies that writhe values contributed by
Natural classification of knots

Figure 2. Opposite. The evolutionary tree of knots. To give a better insight into the structure of the tree, the branches leading to the formation of 6 and 8 crossing knots (lower tree) have been presented independently from branches leading to the formation of 5, 7 and 9 crossing knots (upper tree), whereas the main trunk with unknot, trefoil and figure-eight knot is presented twice. The vertical displacement from the origin (unknot) is proportional to the minimal crossing number (K) of the represented knots, while the horizontal displacement from the origin is proportional to the writh of ideal geometric configurations of the represented knots. Alternating knots that can be converted into each other by one intersegmental passage, and that show a crossing difference of 2, can only differ in their writh by 20/7 (red branches) or 8/7 (blue branches) depending whether the created or eliminated crossings are parallel or antiparallel, respectively. A corresponding passage between non-alternating and alternating knots showing a crossing difference of 2 leads to the writh change of about 2 (green branches). Note that there are groups of knots that have the same writh and the same number of crossings. Coloured squares close to the respective knot notations inform which knots that have two fewer crossings can be obtained by one intersegmental passage from a given knot; the size of the squares provides information about the relative order of kinship of a given knot with its predecessors (see main text for details). Boxed symbolic legend depicts the changes of the writhes introduced by two additional crossings of parallel, antiparallel or of a mixed character.

different types of crossings simply sum up to the final writh of the entire knot. Quantization of writhes is based on the observation that crossings constituting a knot or a link introduce just four quantized values of writhes: +4/7, −4/7, +10/7 and −10/7 that depend on the character (parallel or antiparallel) and the sign of the crossing (Stasiak 2000; see figure 1 for the explanations concerning the signs and characters of crossings). Therefore, the writh value of a given knot provides the information about the types of crossings that constitute a given knot. For this reason, we decided to arrange the knots on their evolutionary tree according to their number of crossings (ordinate value) and their writh value (abscissa).

Figure 2 presents the evolutionary tree for knots with up to 9 crossings. To allow us better insight into the structure of the tree, we have 'split' it into two weakly interconnected parts: the first leading to knots with 6 and 8 crossings (lower part) and the second presenting positions of knots with 5, 7 and 9 crossings (upper part). To avoid repetitions, we have presented only the central and right half of the tree that together include achiral knots and right-handed enantiomorphs of chiral knots with up to 9 crossings. The left half of the tree with left-handed enantiomorphs is simply the mirror image of the right half; therefore, it is not shown, with the exception of left-handed enantiomorphs of the knots 61 and 77. The branches of the tree drawn as red lines in figure 2 connect these knots that can be converted into each other by intersegmental passages which remove (going down the branch) or introduce (going up the branch) parallel crossings into the minimal diagram of a given knot (see figures 1 and 3 for the explanations of parallel (P) and antiparallel (A) crossings). The branches drawn in blue correspond to these intersegmental passages that remove or introduce antiparallel crossings into a minimal diagram of a given knot. Branches leading from left down to right up correspond to intersegmental passages that introduce crossings of positive sign, while branches inclined toward the left indicate intersegmental passages introducing negative crossings. Note, though, that a drawn branch connecting two groups of knots indicates that only some knots in these two groups can be converted into each other by one intersegmental passage of a given type. To find out which knots form predecessor-successor pairs, one needs to use the colour code applied in figure 2. To explain it, let us take an
A. Flammini and A. Stasiak

Figure 3. Intersegmental passages in a minimal crossing diagram of knot $9_9$ which consists of three topological domains. Intersegmental passage at any crossing within a given domain (enclosed within individual ovals) results in a conversion to the same simpler knot. Passages eliminating two parallel or two antiparallel crossings of positive sign decrease the writhe by $20/7$ and $8/7$, respectively. Positive parallel and antiparallel double-arc regions are marked as $+P$ and $+A$, respectively, the respective crossings are marked as $P^+$ and $A^+$.

e example of the $9_9$ knot and its predecessors (figure 3). Analysing random intersegmental passages occurring in random configurations of $9_9$ knots composed of 32 freely jointed segments, we observed that out of more than 14 000 such passages leading to the creation of simpler knots, 42.5% resulted in a conversion into the $7_3$ knot. Conversion into the $7_5$ and $7_1$ knots were observed in 35.7 and 21.8% of the corresponding cases, respectively. To reflect these results in figure 2, the knot $9_9$ (in the penultimate group of 9 crossing knots) is marked with three coloured squares of different sizes. The biggest square is yellow, indicating that the $9_9$ knot most frequently relaxes to the $7_5$ knot (placed on a yellow background field in figure 2). The second biggest square has the same colour as the background field of $7_3$ and the third square in respect of size has the colour of the background field of the $7_1$ knot. Note that the colour code is applied to all knots with seven or more crossings that have predecessors with two fewer crossings and serves two functions: it informs which of the knots with two fewer crossings are indeed the predecessors of a given knot and it provides information about the order of relatedness of a given knot with these predecessors in case there are more of them. Achiral knots (these with zero writhe) undergo random passages to right- and left-handed predecessors with equal probability, but only right-handed predecessors are presented in the figure. Several knots do not have the predecessors with two fewer crossings. In such a case, we indicate what their most complex predecessors are. Thus, knots $9_{38}$ and $9_{40}$ have knot $8_{21}$ as their predecessor, and this is also indicated with the colour code. Knots $8_{18}$ and $9_{31}$ have their predecessors ($9_{31}$ and $6_3$, respectively) marked with the corresponding letters ‘a’ and ‘b’. In figure 2, we did not include the actual conversion rate to different predecessors obtained in the simulation, but reported only the relative order of relatedness to the respective predecessors observed in the simulations. In this respect, it is interesting to mention the case of the knot $8_6$.
for which we analysed over 14,000 passages leading to a simplification in knots topology, and out of those 34.8% led to a passage into the $6_1$ knot and the same percentage led to the $6_2$ knot. Therefore, in figure 2, the $8_6$ knot is marked with two differently coloured squares of the same size.

(b) Minimal diagrams of alternating knots reveal their relatedness with all predecessors

Returning to the example of the $9_9$ knot, it is important to mention that out of 20,000 analysed random passages, all those that led to a conversion to simpler knots (with smaller number of crossings than 9) resulted either in the creation of $7_1$, $7_3$ or $7_5$ knots. Figure 3 shows that the minimal diagram of the knot $9_9$ allows us to predict not only all the possible predecessors of this knot, but also the relative kinship of this knot to its predecessors. If one takes nine minimal diagrams of knot $9_9$ and performs a strand passage at a different crossing in each diagram, then, in four out of nine cases (44.5%), the resulting knot will be $7_5$, in three cases (33.3%) the $7_3$ knot is created and the $7_1$ knot is created in two cases (22.2%). Note a close correspondence between the conversion rates to three different predecessor knots resulting from simulated random passages in random configurations of the $9_9$ knot (see §3a), and these resulting from a simple analysis of the standard, minimal diagram of the $9_9$ knot. In fact, in all the analysed alternating knots, we observed that their minimal diagrams allowed us not only to predict all their predecessor knots (not only those with two fewer crossings), but also to closely estimate the rate of random passages into all predecessor knots of a given successor knot. As could be expected, based on the analysis of random passages, the minimal diagram of the $8_6$ knot has an equal number of crossings that lead to a passage into the $6_1$ and $6_2$ knots. We conjecture that for sufficiently long chains, the proportions between different predecessor knots obtained by random passages will correspond to corresponding proportions that can be predicted based on the minimal diagram of a given alternating knot.

Recall that the three-dimensional writhe value of statistical ensembles of random knots of a given type can be accurately predicted just from the minimal diagrams of alternating knots (Cerf & Stasiak 2000). Our observation concerning random passages provides another example that minimal diagrams of alternating knots capture the important information about the statistical behaviour of the corresponding random knots.

(c) Non-alternating knots

The branches of the tree of knots drawn as green lines in figure 2 correspond to intersegmental passages leading to the creation or relaxation of non-alternating knots. The writhe differences between ideal geometric forms of alternating knots with $n$ crossings and their non-alternating successor knots with $n+2$ crossings are about 2 or $-2$, depending on the sign of introduced crossings (Pieranski 1998). This difference of writhe corresponds therefore to the sum of contributions of parallel and antiparallel crossings of the same sign (10/7 and 4/7; see inset in figure 2). Consequently, to calculate the expected writhe of non-alternating knot from its minimal diagram, we associate a change of writhe of 2 or $-2$ to the passage that eliminates two crossings (of positive or negative sign) and leads to the minimal crossing diagram of an alternating knot with two fewer crossings.
Figure 4. The number and character of crossings in each topological domain is conserved between different minimal crossing diagrams of the same alternating knot. Two different minimal crossing diagrams of knot $7_5$ (a,b) that differ by the flype move have the same number of crossings in each topological domain (crossings encircled with the corresponding colours). Individual crossings are marked with the notations of knots that are created by a strand passage at the corresponding crossing. (c) Topological domains in the abstraction of their connectivity, together with the notations of knots that are created by a strand passage within the domain.

(d) Topological domains

While the disposition of knots according to their minimal crossing number and writhe allowed us to arrange knots into different groups and conveniently present their relatedness, it is necessary to distinguish the knots within groups that have the same writhe and the same crossing number. To this aim, we ‘decompose’ the knots not only into their elementary elements (parallel and antiparallel crossings of two possible signs), but also into higher composing elements that we call topological domains. All crossings belonging to a given topological domain are topologically equivalent to each other, and therefore a topological operation like an intersegmental passage performed at any of the crossings belonging to a given topological domain has the same topological consequence. Figure 3 shows that the right-handed $9_9$ knot is composed of three topological domains. There is a domain with four right-handed parallel crossings, and a first intersegmental passage occurring at any of these crossings leads to the formation of the $7_5$ knot. Then, there is a domain with three parallel right-handed crossings, and the first passage at any of these crossings leads to the formation of the $7_3$ knot. Finally, there is a domain with two right-handed antiparallel crossings, and the first intersegmental passage at any of these crossings leads to the formation of the $7_1$ knot. In the case of the $9_9$ knot drawn in figure 3, the crossings belonging to a
Natural classification of knots

Table 1. Prime knots with up to 7 crossings and their crossing redistribution into parallel (P) and antiparallel (A) topological domains. Positive or negative numbers indicate positive or negative crossings of a given type. Each asterisk indicates that one crossing within a given domain changes its type (from A to P, or contrary) but not the sign during progressive simplification of a knot. Crossings within individual topological domains are topologically equivalent to each other. Brackets group together identical topological domains.

<table>
<thead>
<tr>
<th>knot type</th>
<th>3₁</th>
<th>4₁</th>
<th>5₁</th>
<th>5₂</th>
<th>6₁</th>
<th>6₂</th>
<th>6₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>3*</td>
<td>5*</td>
<td>2</td>
<td>3*</td>
<td>2, -2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>2, -2</td>
<td>3</td>
<td>4, -2</td>
<td>1, -2</td>
<td>1, -1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>knot type</td>
<td>7₁</td>
<td>7₂</td>
<td>7₃</td>
<td>7₄</td>
<td>7₅</td>
<td>7₆</td>
<td>7₇</td>
</tr>
<tr>
<td>P</td>
<td>7*</td>
<td>2</td>
<td>4</td>
<td>3*, 2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>5</td>
<td>3</td>
<td>(3*, 3*), 1</td>
<td>2</td>
<td>2, 1, -2</td>
<td>2, 2, -1, -1, -1</td>
<td></td>
</tr>
</tbody>
</table>

given topological domain form an uninterrupted row of double-arc regions on a minimal crossing diagram of this knot, giving a situation where topological domains are each composed of just one structural domain. This is not always the case. Figure 4 presents two different minimal diagrams of the 7₅ knot that can be converted into each other by the so-called flype move. Although flype moves (Adams 1994) can separate topological domains composed of uninterrupted rows of double-arc regions into two (or more) of such regions, this does not change the number of crossings belonging to each topological domain.

Frequently, knots with symmetrical diagrams have two or more identical topological domains. The topological difference between one or more domains of a given type can be well illustrated by the effect of two successive orientation-preserving smoothings performed on otherwise unchanged diagrams of a knot. Two such smoothings that are performed within the same antiparallel domain always lead to the creation of a disjoint diagram, while this is not the case when the two smoothings are redistributed into two identical but separate antiparallel domains (see figure 1a of electronic supplementary material).

Characters of topological domains (parallel or antiparallel) are defined with respect to the first passage occurring within a minimal diagram of a knot. During progressive simplification of knot diagrams, individual crossings can change their character from parallel to antiparallel (or contrary), but not the sign (figure 3, left). Domains containing such individual crossings are marked with asterisks in table 1 that lists the decompositions of alternating prime knots with up to 7 crossings into their parallel (P) and antiparallel (A) domains of different signs (in the electronic supplementary material, we have included a table listing topological domains of knots with up to 9 crossings). The numerical entries placed in the rows P or A in table 1 (and in a bigger table in the electronic supplementary material) indicate the number of crossings in composing parallel or antiparallel domains, respectively. The signs tell whether crossings in a given domain are positive or negative. Each asterisk, appending a given domain, indicates that one crossing within this domain changes the type but not the sign during the progressive simplification of the knot.

As previously mentioned, there are several possible minimal diagrams of a given knot. However, for all alternating knots analysed here, we did not observe any cases where different minimal diagrams of the same knot had different numbers of crossings.
in the corresponding topological domains. Unfortunately, the constancy of
topological domains does not hold for minimal crossing diagrams of non-alternating
knots. Each of these knots can have numerous different minimal crossing diagrams
that differ in the number of crossings belonging to a given topological domain.
However, all 8 crossing non-alternating knots $8_{19}$, $8_{20}$ and $8_{21}$ have minimal crossing
diagrams that can give rise to all of their predecessor knots upon strand passages
occurring at the crossing points of these diagrams. These diagrams for knots $8_{20}$ and
$8_{21}$ are different from the standard ones in the tables of knots (Rolfsen 1976; Adams
1994); therefore, we have presented them together with the indications of the knot
types resulting from passages at each crossing (see figure 2 of electronic
supplementary material). Alternating knots frequently have minimal crossing
diagrams, where the crossings belonging to every topological domain composed of
two or more crossings are grouped together into an uninterrupted row of double-arc
fields, forming one structural domain (figure 3). In fact, if such a diagram exists for a
given knot, it is usually presented in the standard tables of knots (Rolfsen 1976;
Adams 1994). The entries in table 1 (and its extended version in the electronic
supplementary material) are based on the analysis of minimal diagrams from such
tables with the exception of non-alternating knots $8_{20}$ and $8_{21}$, for which we have used
the diagrams shown in the electronic supplementary material. The type of the domain
(parallel or antiparallel) is easy to define for domains composed of double-arc regions
(figure 3). However, the assignment of the domain type for topological domains
composed of crossings that do not belong to double-arc regions is more complex. In
the electronic supplementary material, a more thorough explanation is provided how
one can decide whether crossings not belonging to double-arc regions have parallel or
antiparallel character.

Analysis of table 1 (and its extended version in the electronic supplementary
material) reveals some simple rules of knots composition in the case of
alternating knots, which are as follows.

(i) All knots have at least one antiparallel crossing for each represented sign.
(ii) Parallel crossings exist as pairs, i.e. if a knot has parallel crossings, it has
an even number of them, although some crossings may only reveal their
real character (parallel or antiparallel) during the progressive simplifica-
tion of the diagram.
(iii) For all knots with up to 9 crossings, their decomposition into topological
domains was unique.

(e) Knots' business cards

Although table 1 (and its extended version in the electronic supplementary
material) allows us to distinguish different knots based on the decomposition of
their minimal diagram, such tables do not inform us about the consequence of
strand passages in each of the topological domains. Therefore, such tables do not
convey the information about the kinship of a given knot with all its
predecessors. Figure 5 presents, in a somewhat abstract way, several simple
knots where crossings constituting topological domains are 'extracted' and
placed in an arbitrary manner not reflecting the actual arrangement and connec-
tivity of the domains in the minimal diagrams. Interestingly, this representation
seems to maintain the essential information about the represented knots.
Figure 5. Knots’ business cards. Topological domains constituting a given knot are presented in such a way that the sign and character of constituting crossings is apparent. Domains placed above and below the middle line of each card are composed of the positive and the negative crossings, respectively. In addition, the applied colour helps to distinguish the positive crossings from the negative ones. Notations placed on the right of each domain indicate the knot type that is created by a strand passage involving any crossing of a given domain. Notations of links placed to the left of each domain indicate link types that are created by smoothing (elimination) of any crossing from a given topological domain. The notation in the upper left corner provides the information about the decomposition of the knot into its domains and crossings, and it allows, for example, to tell what is the expected writhing of this knot. The writhing value of the knot is placed in the upper right corner in the units of 4/6. Nd, the number of topological domains; Nt, the number of topological outcomes resulting from smoothing operations; Ar, the average relaxation value, i.e. the average decrease of the minimal crossing number resulting from smoothing (left) or passage (right). In the accompanying electronic supplementary material, business cards of more complex knots are presented.

The topological domains presented in the upper (positive) parts of the respective panels are composed of positive crossings (the drawn direction arrows form positive crossings; figure 1). The topological domains drawn in the lower (negative) parts of the panels are the negatives of the positive crossings (in literal and mathematical sense). In parallel domains composed of more than one crossing, the direction arrows run in the same direction, while the opposite is the case for antiparallel domains. Crossings represented as ‘vertical’ rectangles correspond to these crossings that contribute values $\pm 10/7$ to the writhing of the knot, depending on the sign of the crossing. Horizontal rectangles correspond to these crossings that contribute $\pm 4/7$ to the writhing of the knot. Domains composed of vertical and horizontal rectangles indicate that the character of one
of the crossings reveals itself only during further steps of progressive simplification of the knot. The number and character of all crossings in each topological domain of a given knot is thus characterized. The knot types denoted to the right of every topological domain indicate the predecessor knots that are created by the strand-passage reaction (sign inversion) at any of the crossings in a given topological domain. The relative ratio between the numbers of crossings in different topological domains determines the kinship order of a given successor knot to its predecessors. In contrast to figure 2, that, for the reasons of simplicity, traced mainly the kinship between the knots that differ in their number of crossings by 2, the panels representing individual knots list all simpler (predecessor) knots that can be created out of the represented knot by an individual segment–segment passage. We have checked that all the predecessor knots were consistent with the tables listing which knots can be converted to each other by one intersegmental passage (Darcy & Sumners 1997; Darcy & Sumners 2000). The link type denoted on the left-hand side of a given topological domain indicates what type of links is created by orientation-preserving smoothing of any of the crossings belonging to a given topological domain (see figure 1 of the electronic supplementary material for examples of orientation-preserving smoothing operations). In most of the cases, it is not necessary to resort to checking the consequences of the orientation-preserving smoothing to determine which crossings belong to the same topological domain. However, in the case of the $4_1$ knot, a strand passage at any of the four crossings leads to the formation of unknot, and this may be interpreted as if all the crossings belong to the same topological domain. In contrast to this, the orientation-preserving smoothing performed at any of the positive antiparallel crossings leads to the creation of left-handed Hopf link, while right-handed Hopf link is created by smoothing performed at any of the negative antiparallel crossings. Therefore, the orientation-preserving smoothing can be used to unambiguously determine which crossings belong to the same topological domain. The numbers in the upper right-hand corner of every panel give the predicted writhe values of the represented knots (i.e. the writhe of ideal geometric representation and the mean writhe of statistical ensemble of random knots of a given type). To operate with integer numbers, the writhe is given in convenient units of $4/7$. In the upper left-hand corners, we propose a notation of individual knots. This notation is similar to that applied in table 1, and it conveys the information about the decomposition of individual knots into composing crossings and topological domains. The capital ‘A’ and ‘P’ letters denote topological domains with antiparallel and parallel apparent character of composing crossings and with all crossings of positive sign. The lower-case letters ‘a’ and ‘p’ denote the corresponding topological domains with negative crossings. The asterisks stand for the crossings that change their character during the progressive simplification of the knot. Brackets group together the structural domains that constitute a given topological domain, but cannot form a continuous double-arc region on a minimal crossing diagram of the knot. In the lower part, we have listed the number of topological domains (Nd) and different topological outcomes (Nt) resulting from a smoothing of crossings in the minimal diagram. We have also listed the average number of crossings the knot loses after a random strand-passage (down right) or smoothing reaction (down left) occurring with equal probability at each crossing of a standard diagram of a given knot. We call this
number the average relaxation value (Ar). Nd, Nt and Ar seem to be invariants for alternating knots and can be easily determined on any minimal crossing diagram of a given alternating knot.

Although a minimal diagram of a knot is its unique determinant, it could be compared to a picture of a person that at sufficient resolution is also its unique determinant. However, in our professional contacts, we prefer to operate with business cards as they convey more relevant information about a given person than its picture. The panels in figure 5 constitute business cards of knots and convey the information about all crossings building a given knot, the number of topological domains in a knot, the relatedness to the predecessors of this knot, etc. In the electronic supplementary material (figures 3–10), we have placed business cards of more knots, including all prime alternating knots with up to 9 crossings. For a better readability, we replaced the parallel crossings by square symbols and the antiparallel ones by circles and noted only the types of knots created by the strand passages within each topological domain.

4. Conclusions

We have presented the new way of classifying all the knot types that takes into account the natural relationship between various knot types resulting from random intersegmental passages within knots with up to 9 crossings.

This work was supported by Swiss National Science Foundation grants 3152-068151 and 3100A0-103962. We thank Profs Isabel Darcy, Klaus Ernst, Eric Rawdon and Mariel Vazquez for their comments on the manuscript.

References

A. Flammini and A. Stasiak


Alessandro FLAMMINI1 and Andrzej STASIAK2*

1 Indiana University, 901E, 10th street, Bloomington, IN 47408, USA

2 University of Lausanne, Lausanne CH 1015, Switzerland

*Author for correspondence (Andrzej.Stasiak@unil.ch).

The electronic supplementary material is available at http://dx.doi.org/10.1098/rspa.2006.1782 or via http://www.journals.royalsoc.ac.uk.