Lattice Stick Numbers of Knots

Youngsik HUH

Mathematical Subject Classification 2000: 57M25, 57M27
Keywords: knot, stick number, cubic lattice

A *polygonal knot* is a simple closed curve in $\mathbb{R}^3$ obtained by joining finitely many points with line segments, called *sticks*. A natural question concerning a polygonal knot is the number of sticks required for its construction. Since the polygonal representation of knots has been considered to be a reasonable model to investigate the knottedness of molecular chains, the number of stick is a meaningful quantity in related topics [11]. According to the restrictions on the positions of the sticks, there are several versions of definitions as listed below.

- *stick number* : $s(K)$ = minimal number of sticks required to construct a polygonal representation of the knot K in $\mathbb{R}^3$.
- *lattice stick number* : $s_L(K)$ = minimal number of sticks required to construct a polygonal representation of the knot K in the cubic lattice $\mathbb{Z}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R})$.
- *lattice edge number* : $e_L(K)$ = minimal number of unit length edges required to construct a polygonal representation of the knot K in $\mathbb{Z}^3$.

Note that a stick consists of one or more unit length edges. For some knots, the stick number and lattice edge number were determined. In particular, $s(3_1) = 6$, $s(4_1) = 7$, $s(K) = 8$ for any knot K of five or six crossings and $s(K) \geq 8$ for all other non-trivial knots K [1, 9, 10]. $e_L(3_1) = 24$ and $e_L(K) > 24$ for all other knots [3, 4].

In this talk we briefly introduce a recent work on knots with small lattice stick numbers.

Theorem 1 *$s_L(3_1) = 12$ and $s_L(K) \geq 14$ for any other non-trivial knot K.*

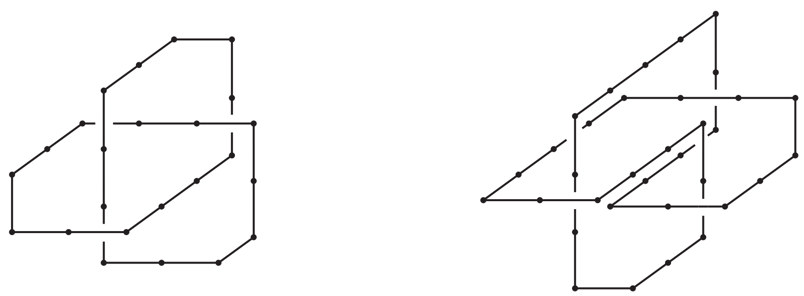
Theorem 2 *$s_L(K) = 14$ if and only if K is 4_1 knot.*

The theorems imply that 3_1 and 4_1 are only knots with lattice stick number up to 14. Some polygonal representations of 3_1 with 12 sticks and 4_1 with 14 sticks are depicted in Figure 1.

To prove the theorems we need a key lemma. $|P|$ denotes the number of sticks of a polygon P. A stick in P which is parallel to the x-axis is called an *x-stick* of P, and $|P|_x$ denotes the number of its x-sticks. Each y-stick or z-stick lies on a plane whose x-coordinate is some integer k. This plane is called an *x-level* k. A polygon P is said to be *properly leveled with respect to the x-coordinate* if each x-level contains exactly two endpoints of x-sticks, and *properly leveled* if it is properly leveled with respect to each coordinate. Note that a properly leveled polygon P has $|P|_x$ x-levels, $|P|_y$ y-levels and $|P|_z$ z-levels.

Lemma 3 *For a polygon P, there is a properly leveled polygon P' equivalent to P so that $|P'| = |P|$ (indeed, the numbers of x, y and z-sticks have remained unchanged).*

The properly leveledness induces several constraints on the number and positions of x-sticks(also, y, z-sticks). For example, if P is a properly leveled non-trivial polygon then $|P|_x$, $|P|_y$ and $|P|_z$ should be at least 4, which implies $s_L(3_1) = 12$. If $|P| \leq 14$, then $(|P|_x, |P|_y, |P|_z)$ is one of $(4, 4, 4)$, $(5, 4, 4)$, $(6, 4, 4)$ and $(5, 5, 4)$. The constraints from the properly leveledness enable us to list all possible paths of the projection of P onto one y-level. After checking each path, we can prove that only 3_1 or 4_1 are obtained from the projection. The details of the proof can be found in [5] and [6].



(a) (b)

Figure 1: 3_1 and 4_1 in the cubic lattice

References

[1] C. Adams, *The Knot Book*, W.H. Freedman & Co., New York, 1994.

[2] Colin C. Adams, Belvin M. Brennan, Deborah L. Greilsheimer and Alexander K. Woo, *Stick numbers and composition of knots and links*, J. Knot Theory Ramif. **6** (1997) 149–161.

[3] Y. Diao, *Minimal knotted polygons on the cubic lattice*, J. Knot Theory Ramif. **2** (1993) 413–425.

[4] Y. Diao, *The number of smallest knots on the cubbic lattice*, J. Stat. Phys. **74** (1994) 1247–1254.

[5] Y. Huh and S. Oh, *Lattice stick numbers of small knots*, J. Knot Theory Ramif. **14** (2005) 859–867.

[6] Y. Huh and S. Oh, *Knots with lattice stick number upto 14*, Preprint.

[7] E. J. Janse Van Rensburg and S. D. Promislow, *Minimal knots in the cubic lattice*, J. Knot Theory Ramif. **4** (1995) 115–130.

[8] E. J. Janse Van Rensburg and S. D. Promislow, *The curvature of lattice knots*, J. Knot Theory Ramif. **8** (1999) 463–490.

[9] S. Negami, *Ramsey theorems for knots, links, and spatial graphs*, Trans. Amer. Math. Soc. **324** (1991) 527–541.

[10] R. Randell, *An elementary invariant of knots*, J. Knot Theory Ramif. **3** (1994) 279–286.

[11] D. W. Sumners and S. G. Whittington, *Knots in self-avoiding walks*, J. Phys. A: Math. Gen. **21** (1988) 1689–1694.

Department of Mathematics, College of Natural Sciences, Hanyang University, Seoul 133-791, Korea
yshuh@hanyang.ac.kr