CLASSIFICATION OF POLYNOMIAL SEMIGROUPS AND RANDOM JULIA SETS
THAT ARE JORDAN CURVES BUT NOT QUASICIRCLES

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ABSTRACT. We investigate the dynamics of semigroups generated by a family of polynomial maps on the Riemann sphere \( \hat{C} \) and random dynamics of polynomials. We classify compactly generated (semi)-hyperbolic semigroups with bounded planar postcritical set. We show that there are three classes among such semigroups, and in one of the classes, for almost sure sequence \( \gamma \), the random Julia set \( J_\gamma \) of \( \gamma \) is a Jordan curve but not a quasicircle, and the unbounded component of \( \hat{C} \setminus J_\gamma \) is a John domain. We show that there exists such a semigroup. Note that this phenomenon does not occur in the usual dynamics of a single polynomial.

1. INTRODUCTION AND THE MAIN RESULTS

In this paper, we investigate a special phenomenon which can hold in the random dynamics of a family of polynomials, but cannot hold in the usual iteration dynamics of a single polynomial.

In order to state the main results, we give some notations and definitions.

Definition 1.

- We set \( Y := \{ g : \hat{C} \to \hat{C} \mid g \text{ is a polynomial, } \deg(g) \geq 2 \} \) endowed with the topology induced by the uniform convergence on the Riemann sphere \( \hat{C} \).
- We set \( Y^N := \{ (\gamma_1, \gamma_2, \ldots) \mid \forall j \in \mathbb{N}, \gamma_j \in Y \} \) endowed with the product topology.
- For each \( \gamma = (\gamma_1, \gamma_2, \ldots) \in Y^N \), we set \( F_\gamma := \{ z \in \hat{C} \mid \{ \gamma_n \circ \cdots \circ \gamma_1 \}_{n \in \mathbb{N}} \text{ is normal around } z \} \).
- For each \( \gamma \in Y^N \), we set \( J_\gamma := \hat{C} \setminus F_\gamma \). This is called the random Julia set of \( \gamma \).
- Let \( \gamma \in Y^N \). If \( \infty \in F_\gamma \), we denote by \( A_\gamma \) the connected component of \( F_\gamma \) containing \( \infty \).

Remark 1. If \( \Gamma \) is a non-empty compact subset of \( Y \) and \( \gamma \in \Gamma^N \), then \( \infty \in F_\gamma \).

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Definition 2. A polynomial semigroup $G$ is a semigroup generated by a family of non-
constant polynomial maps on $\hat{\mathbb{C}}$ with the semigroup operation being functional com-
position. For a polynomial semigroup $G$, we use the following notation.

- We set $F(G) := \{z \in \hat{\mathbb{C}} \mid G \text{ is normal around } z\}$. This is called the Fatou set of $G$.
- We set $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This is called the Julia set of $G$.
- We set $P(G) := \bigcup_{g \in G} \{\text{all critical values of } g : \mathbb{C} \to \mathbb{C}\} \subset \hat{\mathbb{C}}$. This is called the postcritical set of $G$. Moreover, we set $P^*(G) := P(G) \setminus \{\infty\}$.
- Let $h \in \mathcal{Y}$. We set $J(h) := J(\langle h \rangle)$ and $F(h) := F(\langle h \rangle)$. Moreover, we denote by $K(h)$ the filled-in Julia set of $h$.

Definition 3. Let $G$ be a polynomial semigroup.

- $G$ is said to be hyperbolic if $P(G) \subset F(G)$.
- $G$ is said to be semi-hyperbolic if for each $x \in J(G)$, there exists a number $\delta > 0$ and a positive integer $N$ such that for each $g \in G$, $\deg(g : V \to B(x, \delta)) \leq N$ for each component $V$ of $g^{-1}(B(x, \delta))$, where $B(x, \delta)$ denotes the spherical ball with center $x$ and radius $\delta$.

Definition 4. Let $K \geq 1$. A Jordan curve $\xi \subset \mathbb{C}$ is said to be a $K$-quasicircle if there exists a $K$-quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(\{z \in \mathbb{C} \mid |z| = 1\}) = \xi$.

Definition 5. Let $V$ be a subdomain of $\hat{\mathbb{C}}$ such that $\partial V \subset \mathbb{C}$. We say that $V$ is a John domain if there exists a constant $c > 0$ and a point $z_0 \in V$ ($z_0 = \infty$ when $\infty \in V$) satisfying the following: for all $z_1 \in V$ there exists an arc $\xi \subset V$ connecting $z_1$ to $z_0$ such that for any $z \in \xi$, we have $\min\{|z - a| \mid a \in \partial V\} \geq c|z - z_1|$.

Definition 6. Let $X$ be a complete metric space. A subset $U$ of $X$ is said to be residual if $U$ contains a countable intersection of open dense subsets of $X$.

We now state the main results of this paper (Theorem A and Theorem B). These results and the details of the proofs are included in [5].

Theorem A. Let $\Gamma$ be a non-empty compact subset of $\mathcal{Y}$. Let $G$ be the polynomial semigroup generated by $\Gamma$. Suppose that $P^*(G)$ is bounded in $\mathbb{C}$ and $G$ is semi-hyperbolic. Then, exactly one of the following three statements (I), (II), (III) holds.

(I) $G$ is hyperbolic. Moreover, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^N$, $J_\gamma$ is a $K$-quasicircle.

(II) There exists a residual subset $U$ of $\Gamma^N$ such that for each Borel probability measure $\tau$ in $\mathcal{Y}$ with $\text{supp } \tau = \Gamma$, we have $(\otimes_{i=1}^{\infty} \tau)(U) = 1$. and such that for each $\gamma \in U$, $J_\gamma$ is a Jordan curve but not a quasicircle, $A_\gamma$ is a John domain, and the bounded component of $F_\gamma$ is not a John domain. Moreover, there exists a dense subset $\mathcal{V}$ of $\Gamma^N$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma$ is not a Jordan curve. Furthermore, there exist two elements $\alpha, \beta \in \Gamma^N$ such that $J_\beta$ is included in a bounded component of $\mathbb{C} \setminus J_\alpha$.

(III) There exists a dense subset $\mathcal{W}$ of $\Gamma^N$ such that for each $\gamma \in \mathcal{W}$, $J_\gamma$ is not a Jordan curve. Moreover, for each $\alpha, \beta \in \Gamma^N$, $J_\alpha \cap J_\beta \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.
Theorem B. Let $\Gamma$ be a non-empty compact subset of $\mathbf{Y}$. Let $G$ be the polynomial semigroup generated by $\Gamma$. Suppose that $P^*(G)$ is bounded in $\mathbb{C}$ and $G$ is hyperbolic. Then, exactly one of the following three statements (I), (II), (III) holds.

(I) There exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^N$, $J_\gamma$ is a $K$-quasicircle.

(II) There exists a residual subset $U$ of $\Gamma^N$ such that for each Borel probability measure $\tau$ in $\mathbf{Y}$ with $\text{supp } \tau = \Gamma$, we have $(\bigotimes_{j=1}^{\infty} \tau)(U) = 1$, and such that for each $\gamma \in U$, $J_\gamma$ is a Jordan curve but not a quasicircle, $A_\gamma$ is a John domain, and the bounded component of $F_\gamma$ is not a John domain. Moreover, there exists a dense subset $V$ of $\Gamma^N$ such that for each $\gamma \in V$, $J_\gamma$ is a quasicircle. Furthermore, there exists a dense subset $W$ of $\Gamma^N$ such that for each $\gamma \in W$, there are infinitely many bounded connected components of $F_\gamma$.

(III) For each $\gamma \in \Gamma^N$, there are infinitely many bounded connected components of $F_\gamma$. Moreover, for each $\alpha, \beta \in \Gamma^N$, $J_\alpha \cap J_\beta \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.

Example 1. Let $g_1(z) := z^2 - 1$ and $g_2(z) := \frac{z^2}{2}$. Let $\Gamma := \{g_1^2, g_2^2\}$. Moreover, let $G$ be the polynomial semigroup generated by $\Gamma$. Then, $P^*(G)$ is bounded in $\mathbb{C}$, $G$ is hyperbolic, and statement (II) in Theorem B holds. (In this example, for each component $J$ of $J(G)$, there exists a unique $\gamma \in \Gamma^N$ such that $J_\gamma = J$. See Figure 1.)

![Figure 1. The Julia set of $G = \langle g_1^2, g_2^2 \rangle$.](image)

Remark 2. Let $h \in \mathbf{Y}$ and suppose that $J(h)$ is a Jordan curve but not a quasicircle. Then the basin of $\infty$ is not a John domain. Thus what we see in statement (II) in Theorem A and statement (II) in Theorem B, as illustrated in the above Example, is a special phenomenon which can hold in the random dynamics of a family of polynomials, but cannot hold in the usual iteration dynamics of a single polynomial.

2. Tools

In this section, we give some tools to prove the main results.

Definition 7. Let $\Gamma$ be a non-empty compact subset of $\mathbf{Y}$. Let $\sigma : \Gamma^N \to \Gamma^N$ be the shift map, which is defined by $\sigma(\gamma_1, \gamma_2, \ldots) = (\gamma_2, \gamma_3, \ldots)$. We define a map $f : \Gamma^N \times \mathbb{C} \to \Gamma^N \times \mathbb{C}$ by: $(\gamma, y) \mapsto (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^N$. This is called the skew product associated with the family $\Gamma$. Moreover, we use the following notation.
Lemma 1. Let $\Gamma$ be a non-empty compact subset of $\mathbb{Y}$. Let $f : \Gamma^N \times \hat{\mathbb{C}} \to \Gamma^N \times \hat{\mathbb{C}}$ be the skew product associated with $\Gamma$. Then, we have the following.

(I) ([3, Lemma 2.4]) For each $\gamma \in \Gamma^N$, $(f_{\gamma,1})^{-1}(J_{\sigma(\gamma)}) = J_\gamma$. Furthermore, we have $\hat{J}_\gamma = J_\gamma$. Note that equality $\hat{J}_\gamma = J_\gamma$ does not hold in general. Moreover, $f^{-1}(\hat{J}(f)) = \hat{J}(f) = f(\hat{J}(f))$, and for each $\gamma \in \Gamma^N$, $(f_{\gamma,1})^{-1}(\hat{J}_{\sigma(\gamma)}) = \hat{J}_\gamma$.

(II) ([2, 3]) For each $\gamma \in \Gamma^N$, $J_\gamma$ is a non-empty perfect set with $\sharp(J_\gamma) \geq 3$. Furthermore, the map $\gamma \mapsto J_\gamma$ is lower semicontinuous; i.e., for any element $(\gamma, y) \in \Gamma^N \times \hat{\mathbb{C}}$ with $y \in J_\gamma$ and any sequence $\{y^m\}_{m \in \mathbb{N}}$ in $\Gamma^N$ with $y^m \to \gamma$, there exists a sequence $\{y_m\}_{m \in \mathbb{N}}$ in $\Gamma^N$ with $y_m \to y$ for each $m \in \mathbb{N}$ such that $y_m \to y$. However, $\gamma \mapsto J_\gamma$ is NOT continuous with respect to the Hausdorff topology in general.

(III) There exists a ball $B$ around $\infty$ such that for each $\gamma \in \Gamma^N$, $B \subset A_\gamma \subset F_\gamma$. Moreover, for each $\gamma \in \Gamma^N$, $J_\gamma = \partial(K_\gamma) = \partial(A_\gamma)$ and $A_\gamma$ is connected.

(IV) Let $G$ be a polynomial semigroup generated by $\Gamma$. Then, $\pi_G(\hat{J}(f)) = J(G)$.

(V) Let $G$ be a polynomial semigroup generated by $\Gamma$. Then, $P^*(G)$ is bounded in $\mathbb{C}$ if and only if for each $\gamma \in \Gamma^N$, $J_\gamma$ is connected.

(VI) If $\omega \in \Gamma^N$ is an element such that $\text{int}(K_\omega)$ is a non-empty set, then $\overline{\text{int}(K_\omega)} = K_\omega$ and $\partial(\text{int}(K_\omega)) = J_\omega$.

The following is one of the keys to prove the main results.

Theorem 1 ([3]). Let $\Gamma$ be a non-empty compact subset of $\mathbb{Y}$. Let $G$ be the polynomial semigroup generated by $\Gamma$. Suppose $G$ is semi-hyperbolic. Then, $\gamma \mapsto J_\gamma$ is continuous with respect to the Hausdorff topology.

From this theorem, we get the following, which is also one of the keys to prove the main results.

Theorem 2 ([4]). Let $\Gamma$ be a non-empty compact subset of $\mathbb{Y}$. Let $G$ be the polynomial semigroup generated by $\Gamma$. Suppose $G$ is semi-hyperbolic. Moreover, suppose $P^*(G)$ is bounded in $\mathbb{C}$. Then, for each $\gamma \in \Gamma^N$, $A_\gamma$ is a John domain and $J_\gamma$ is connected and locally connected.

3. OUTLINE OF THE PROOFS OF THE MAIN RESULTS

In this section, we give the outline of the proofs of the main results.

Using Theorem 1 and the ‘uniform fiberwise quasiconformal surgery,’ we obtain the following result.
Theorem 3 ([5]). Let $\Gamma$ be a non-empty compact subset of $Y$. Let $G$ be a polynomial semigroup generated by $\Gamma$. Suppose that $G$ is hyperbolic and $P^*(G)$ is bounded in $C$. Moreover, suppose that for each $\gamma \in \Gamma^N$, $\text{int}(K_{\gamma})$ is connected. Then, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^N$, $J_{\gamma}$ is a $K$-quasicircle.

Using Theorem 2, we obtain the following proposition.

Proposition 1 ([5]). Let $\Gamma$ be a non-empty compact subset of $Y$. Let $G$ be a polynomial semigroup generated by $\Gamma$. Suppose that $G$ is semi-hyperbolic and $P^*(G)$ is bounded in $C$. Let $\omega \in \Gamma^N$ be an element. If $\text{int}(K_{\gamma})$ is a non-empty connected set, then $J_{\omega}$ is a Jordan curve.

We give some lemmas to prove the main results.

Definition 8. Let $h \in Y$. Suppose that $J(h)$ is connected. Let $\psi$ be a biholomorphic map $\hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_{\infty}(h)$ with $\psi(\infty) = \infty$ such that $\psi^{-1} \circ h \circ \psi(z) = z^{\deg(h)}$ for each $z \in \hat{\mathbb{C}} \setminus \overline{D(0,1)}$, where $F_{\infty}(h)$ denotes the unbounded component of $F(h)$. For each $\theta \in \partial D(0,1)$, we set $S(\theta) := \psi(\{r\theta \mid 1 < r \leq \infty\})$. This is called the external ray (for $K(h)$) with angle $\theta$. 

Lemma 2 ([5]). Let $h \in Y$. Suppose that $J(h)$ is connected and locally connected and $J(h)$ is not a Jordan curve. Moreover, suppose that there exists an attracting periodic point of $h$ in $K(h)$. Then, for any $\epsilon > 0$, there exist a point $p \in J(h)$ and elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$, such that all of the following hold.

(I) For each $i = 1, 2$, the external ray $S(\theta_i)$ lands at the point $p$.

(II) Let $V_1$ and $V_2$ be the two connected components of $\hat{\mathbb{C}} \setminus (S(\theta_1) \cup S(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J(h) \neq \emptyset$. Moreover there exists an $i$ such that $\text{diam}(V_i \cap K(h)) \leq \epsilon$.

The following is the key lemma to prove the main results.

Lemma 3 ([5]). Let $\Gamma$ be a non-empty compact subset of $Y$. Let $G$ be the polynomial semigroup generated by $\Gamma$. Let $\mu > 0$ be a number. Then, there exists a number $\delta > 0$ such that the following statement holds.

- Let $\omega \in \Gamma^N$ be any element and $p \in J_{\omega}$ any point with $\min\{|p - b| \mid b \in P^*(G)\} > \mu$. Suppose that $J_{\omega}$ is connected. Let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_{\omega}$ be a biholomorphic map with $\psi(\infty) = \infty$. For each $\theta \in \partial D(0,1)$, let $T(\theta) = \psi(\{r\theta \mid 1 < r \leq \infty\})$. Suppose that there exist two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2$, $T(\theta_i)$ lands at $p$. Moreover, suppose that a connected component $V$ of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ satisfies that $\text{diam}(V \cap K_{\omega}) \leq \delta$. Furthermore, let $\gamma \in \Gamma^N$ be any element and suppose that there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that $\sigma^{n_k}(\gamma) \to \omega$ as $k \to \infty$, where $\sigma : \Gamma^N \to \Gamma^N$ denotes the shift map. Then, $J_{\gamma}$ is not a quasicircle.

Using Lemma 1, Theorem 1, Theorem 2, Theorem 3, Proposition 1, Lemma 2, Lemma 3, and some further arguments, we can prove Theorem A and Theorem B.

References


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