RANDOM DYNAMICS OF POLYNOMIALS AND POSTCRITICALLY BOUNDED POLYNOMIAL SEMIGROUPS

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ABSTRACT. We consider the random dynamics of a family of complex polynomial maps \( \{ h_\lambda : \lambda \in \Lambda \} \) on the Riemann sphere \( \hat{\mathbb{C}} \), that is, we consider the dynamics of any iteratively defined composition sequence of maps \( h_{\lambda_n} \circ \cdots \circ h_{\lambda_1} \) where each \( \lambda_k \in \Lambda \). In this paper we concern ourselves with questions of dynamic stability, along such composition sequences one at a time as well as the study of when such stability exists no matter which composition sequence is chosen. We are then naturally led to the study of the dynamics of polynomial semigroups (semigroups generated by a family of polynomial maps) on \( \hat{\mathbb{C}} \). We present several results with regards to the structure of the sets of stability when the corresponding polynomial semigroup has bounded planar postcritical set. Specifically, we investigate how the sets of instability (Julia sets) of the composition sequences are distributed within the set of instability (Julia set) of the entire polynomial semigroup.

This is an expository paper whose main results are presented in detail in [16].

The study of (rational) random dynamics and (rational) semigroup dynamics must first begin with the study of (rational) iteration dynamics. We denote by \( \hat{\mathbb{C}} \) the Riemann sphere. Let \( z_0 \in \hat{\mathbb{C}} \) and \( f(z) \) be a rational map on \( \hat{\mathbb{C}} \). Iteration dynamics is concerned with studying questions about the properties and behavior of the orbit \( \{ f^n(z_0) \}_{n=0}^\infty \) (e.g., is the orbit attracted to a cycle?, is it dense in \( \hat{\mathbb{C}} \)?, is it dense in some other interesting set?) as well as studying the question of stability of the orbit (i.e., will all points near \( z_0 \) have a similar orbit?, i.e., is the family of iterates \( \{ f^n \} \) equicontinuous (with respect to the spherical metric) on some neighborhood of \( z_0 \)?). Traditionally the set of starting points \( z_0 \) which have a neighborhood on which \( \{ f^n \} \) is equicontinuous (equivalently normal) is called the Fatou set of \( f \) and its complement in \( \hat{\mathbb{C}} \) is called the Julia set of \( f \). It is well known that for \( f(z) = z^2 \) the Julia set is the unit circle and for \( f_c(z) = z^2 + c \), where \( |c| \) is small, the Julia set is a quasicircle. However, if \( |c| \) is large, then the Julia set of \( f(z) = z^2 + c \) is a Cantor set. It turns out that the Julia set of the map \( f(z) = z^2 + c \) is connected exactly when the postcritical orbit \( \{ f^n(0) \}_{n=1}^\infty \) is bounded in \( \mathbb{C} \), and otherwise the Julia set is a Cantor set. Hence we see in a striking way the importance of what is called the postcritically bounded condition.

In this paper we will look at a natural generalization of iteration dynamics, where instead of repeatedly applying a single map over and over again, we will allow the map to be changed at each step of the orbit. Assigning probabilities to the choice of map at each stage is the setting for research of random dynamics (see [7, 3, 5, 6, 4, 24, 25]). We
begin by considering a family of generator rational maps \( \{ h_\lambda : \lambda \in \Lambda \} \) where each \( h_\lambda \) is rational with \( \deg(h_\lambda) \geq 2 \). Following F. Ren, Z. Gong, and W. Zhou (see [28, 8]), for each sequence of generator maps \( x = (h_{\lambda_1}, h_{\lambda_2}, h_{\lambda_3}, \ldots) \in \{ h_\lambda : \lambda \in \Lambda \}^N \), we define a Composition Sequence of maps \( S_x = \{ h_{\lambda_1}(z), h_{\lambda_2} \circ h_{\lambda_1}(z), h_{\lambda_3} \circ h_{\lambda_2} \circ h_{\lambda_1}(z), \ldots, h_{\lambda_n} \circ \cdots \circ h_{\lambda_1}(z), \ldots \} \). For each \( x = (h_{\lambda_1}, h_{\lambda_2}, h_{\lambda_3}, \ldots) \in \{ h_\lambda : \lambda \in \Lambda \}^N \), the Fatou set of \( S_x \) is \( F_x = \{ z \in \mathbb{C} : \exists \text{ a neighborhood of } z \text{ on which } S_x \text{ is a normal family} \} \), and the Julia set of \( S_x \) is given by \( J_x = \mathbb{C} \setminus F_x \). Thus \( F_x \) is the set of points which have stability of the orbit along the composition sequence \( S_x \) and \( J_x \) is the set of points which do not.

A point of interest in this study is to identify and describe the set of points which have stability of the orbit along every composition sequence. As we shall see, this leads one to investigate the notion of a rational semigroup. Following A. Hinkkanen and G. J. Martin (see [9]), we denote the Rational Semigroup \( G \) generated by \( \{ h_\lambda : \lambda \in \Lambda \} \) using the notation \( G = \langle h_\lambda : \lambda \in \Lambda \rangle \), where the semigroup operation is composition of functions. We then define the Fatou set of \( G \) by \( F(G) = \{ z \in \mathbb{C} : \exists \text{ a neighborhood of } z \text{ on which } G \text{ is a normal family} \} \) and the Julia set of \( G \) by \( J(G) = \mathbb{C} \setminus F(G) \). Please note that we employ the common notation from iteration theory and write \( F(h) = F(h) \) and \( J(h) = J(h) \).

Remark: One may wish to allow some or all of the maps in \( G \) to be Möbius, for example, when one is considering Kleinian groups as in [13], but since such settings are not relevant to this paper we will use our simplified definition to avoid any technical complications. Thus in this paper, a rational semigroup (resp. a polynomial semigroup) is a semigroup generated by a family of rational maps (resp. polynomial maps) of degree two or more.

This paper is concerned with studying the relationship between properties of the Julia set and the analog of the postcritically bounded condition. The postcritical set of a rational semigroup \( G \) is defined by \( P(G) = \bigcup_{g \in G} \{ \text{all critical values of } g \} \), and the finite/planar postcritical set of \( G \) is defined by \( P^*(G) = P(G) \setminus \{ \infty \} \). We study the Julia sets \( J_g \) and \( J(G) \) under the condition that the rational semigroup of polynomials \( G = \langle h_\lambda : \lambda \in \Lambda \rangle \) has a bounded planar postcritical set, i.e., \( P^*(G) \) is bounded in \( \mathbb{C} \). We denote the class of all such polynomial semigroups by \( \mathcal{G} \) and we partition \( \mathcal{G} \) into the collections \( \mathcal{G}_{\text{dis}} = \{ G \in \mathcal{G} : J(G) \text{ is disconnected} \} \) and \( \mathcal{G}_{\text{con}} = \{ G \in \mathcal{G} : J(G) \text{ is connected} \} \). For a rational map \( f \), we set \( P(f) = P((f)) \). Moreover, for a polynomial map \( f \), we set \( P^*(f) = P^*((f)) \).

Remark: If \( G = \langle h_\lambda : \lambda \in \Lambda \rangle \), then \( P(G) = \bigcup_{\lambda \in \Lambda} \cup_{z \in CV(h_\lambda)} (G(z) \cup \{ z \}) \) where \( CV(h) \) denotes the critical values of \( h \) and \( G(z) \) denotes the forward orbit of \( z \) (i.e., \( G(z) = \cup_{g \in G} g(\{ z \}) \)). From this one may, in the finitely generated case, use a computer to see if \( G \in \mathcal{G} \) much in the same way as one verifies the boundedness of the critical orbit for the maps \( f_c(z) = z^2 + c \). See [14] for a web link to download the program “Julia” which can be used to both check the postcritical boundedness condition as well as draw the Julia set for certain rational semigroups.

Remark: We say that \( G \) is hyperbolic if \( P(G) \subset F(G) \). For research on (semi-) hyperbolicity and Hausdorff dimension of Julia sets of rational semigroups in general see [17, 18, 19, 20, 23, 26]. For related results for Julia sets of semigroups in the class \( \mathcal{G} \) see [25, 16].
It is well known in iteration theory (see [1]) that when \( f \) is a polynomial of degree two or more, \( J(f) \) is connected if and only if \( P^*(f) \) is bounded in \( \mathbb{C} \). Hence, for any \( g \in G \in \mathcal{G} \), we have that \( J(g) \) is connected. Unlike in iteration, however, we may have \( G \in \mathcal{G} \), yet still have \( J(G) \) disconnected (see [27, 25]). See also [22, 25] for an analysis of the number of connected components of \( J(G) \) involving the inverse limit of connected components of the realizations of the nerves of finite coverings \( \mathcal{U} \) of \( J(G) \), where \( \mathcal{U} \) consists of backward images of \( J(G) \) under maps in \( G \).

In general, we wish to know what can be said about the structure of the Julia set and the dynamics of semigroups \( G \in \mathcal{G} \)? We first note that a natural order (that is respected by the backward action of the maps in \( G \)) can be placed on the components of \( J(G) \), which then leads to implications on the connectivity of Fatou components. Before we can state this properly, we provide the following background and notation.

We quote the following results from [9]. The Fatou set \( F(G) \) is forward invariant under each element of \( G \), i.e., \( g(F(G)) \subset F(G) \) for all \( g \in G \), and thus \( J(G) \) is backward invariant under each element of \( G \), i.e., \( g^{-1}(J(G)) \subset J(G) \) for all \( g \in G \). We should take a moment to note that the sets \( F(G) \) and \( J(G) \) are, however, not necessarily completely invariant under the elements of \( G \). This is in contrast to the case of iteration dynamics, i.e., the dynamics of semigroups generated by a single rational function. For a treatment of alternatively defined completely invariant Julia sets of rational semigroups the reader is referred to [10], [11], [12] and [15].

In what follows we employ the following notation. For any polynomial \( g \) with degree greater or equal to 2, we denote the filled-in Julia set of \( g \) by \( K(g) := \{ z \in \mathbb{C} \mid \cup_{n \in \mathbb{N}} g^n(\{ z \}) \} \) is bounded in \( \mathbb{C} \). We note that \( J(g) = \partial K(g) \) and that \( K(g) \) is the polynomial hull of \( J(g) \). The appropriate extension of the concept of the filled-in Julia set is as follows. (See [9, 2] for other kinds of filled-in Julia sets.) For a polynomial semigroup \( G \), we define the smallest filled-in Julia set to be

\[
\hat{K}(G) = \{ z \in \mathbb{C} \mid G(z) \text{ is bounded in } \mathbb{C} \}.
\]

**Notation:** For a polynomial semigroup \( G \in \mathcal{G} \), we denote by \( \mathcal{J} = \mathcal{J}_G \) the set of all connected components of \( J(G) \) which do not include \( \infty \).

**Definition 1.** We place a partial order on the space of all non-empty connected sets in \( \mathbb{C} \) as follows. For connected sets \( K_1 \) and \( K_2 \) in \( \mathbb{C} \). "\( K_1 \leq K_2 \)" indicates that \( K_1 = K_2 \) or \( K_1 \) is included in a bounded component of \( \mathbb{C} \setminus K_2 \). Also, "\( K_1 < K_2 \)" indicates \( K_1 \leq K_2 \) and \( K_1 \neq K_2 \). We call \( \leq \) the surrounding order and read \( K_1 < K_2 \) as "\( K_1 \) is surrounded by \( K_2 \)".

**Theorem A ([21, 25]).** Let \( G \in \mathcal{G} \) (possibly infinitely generated). Then

1. \( (\mathcal{J}, \leq) \) is totally ordered.
2. Each connected component of \( F(G) \) is either simply or doubly connected.
3. For any \( g \in G \) and any connected component \( J \) of \( J(G) \), we have that \( g^{-1}(J) \) is connected. Let \( g^*(J) \) be the connected component of \( J(G) \) containing \( g^{-1}(J) \). If \( J \in \mathcal{J} \), then \( g^*(J) \in \mathcal{J} \). If \( J_1, J_2 \in \mathcal{J} \) and \( J_1 \leq J_2 \), then both \( g^{-1}(J_1) \leq g^{-1}(J_2) \) and \( g^*(J_1) \leq g^*(J_2) \).

We note that under the hypothesis of the above theorem \( J_1 < J_2 \) for \( J_1, J_2 \in \mathcal{J} \) does not imply \( g^*(J_1) < g^*(J_2) \), but only that \( g^*(J_1) \leq g^*(J_2) \) as can be seen in the example given later.
For our remaining results we need to note the existence of both a minimal element and a maximal element in \( \mathcal{J} \) and state a few of their properties.

**Theorem B** ([25]). Let \( G \) be a polynomial semigroup in \( \mathcal{G}_{\text{dis}} \). Then there is a unique element \( J_{\text{min}}(G) \) (abbreviated by \( J_{\text{min}} \)) \( \in \mathcal{J} \) such that \( J_{\text{min}} \) meets (and therefore contains) \( \partial \mathcal{K} \). Also, \( \infty \in F(G) \) and there exists a unique element \( J_{\text{max}}(G) \) (abbreviated by \( J_{\text{max}} \)) \( \in \mathcal{J} \) such that \( J_{\text{max}} \) meets (and therefore contains) \( \partial \mathcal{U}_\infty \), where \( \mathcal{U}_\infty \) is the simply connected component of \( F(G) \) which contains \( \infty \). Furthermore, we have the following

- \( J_{\text{min}} \leq J \) for all \( J \in \mathcal{J} \),
- \( J_{\text{max}} \geq J \) for all \( J \in \mathcal{J} \),
- \( \mathcal{K}(G) \), and therefore \( P^*(G) \), is contained in the polynomial hull of each \( J \in \mathcal{J} \).

We see that \( \partial \mathcal{K}(G) \subset J(G) \) when \( G \in \mathcal{G} \), but, in general, we do not have \( \partial \mathcal{K}(G) = J(G) \), unlike in iteration theory where \( \partial \mathcal{K}(g) = J(g) \) for polynomials \( g \) of degree two or more. In fact, \( \partial \mathcal{K}(G) \) might not even equal \( J_{\text{min}}(G) \) (see [16]).

When \( G \in \mathcal{G}_{\text{con}} \) we will use the convention that \( J_{\text{min}} = J_{\text{max}} = J(G) \). However, it is not necessarily the case that \( \infty \in F(G) \), as exhibited by the example \( \langle z^2/n : n \in \mathbb{N} \rangle \).

Note that \( J(G) \) contains the Julia set of each element of \( G \). Moreover, the following critically important result due to Hinkkanen and Martin holds.

**Theorem C** ([9], Corollary 3.1). For rational semigroups \( G \) with \( \sharp(J(G)) \geq 3 \), we have

\[
J(G) = \bigcup_{f \in G} J(f).
\]

**Remark:** Theorem C can be used to easily show that \( F(\langle h_\lambda : \lambda \in \Lambda \rangle) \) is precisely the set of \( z \in \overline{\mathbb{C}} \) which has a neighborhood on which every composition sequence generated by \( \{h_\lambda : \lambda \in \Lambda \} \) is normal (see [28]).

In the proofs of many results concerning postcritically bounded polynomials semigroups (see, in particular, [25] and [16]), it is critical to understand the distribution of the sets \( J(g) \) where \( g \in G \), especially when \( g \) is a generator of \( G \). In particular, it is important to understand the relationship between such \( J(g) \) and the special components \( J_{\text{min}} \) and \( J_{\text{max}} \) of \( J(G) \). In this paper we investigate such matters carefully.

**Definition 2.** We denote by \( \text{Poly} \) the space of all polynomial maps on \( \overline{\mathbb{C}} \) endowed with the topology induced by the uniform convergence on \( \overline{\mathbb{C}} \).

**Lemma 1.** [25] If \( G \in \mathcal{G} \) is generated by a compact family in \( \text{Poly} \), then both \( J_{\text{min}} \) and \( J_{\text{max}} \) must contain \( J(h_\lambda) \) for some generator \( h_\lambda \in G \).

Let \( G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G} \). We say that \( J \in \mathcal{J} \) has property \((\star)\) if \( J \) contains \( J(g) \) for some \( g \in G \). We say that \( J \in \mathcal{J} \) has property \((\star\lambda)\) if \( J \) contains \( J(h_\lambda) \) for some generator \( h_\lambda \in G \).

**Proposition 1.** [16] Let \( G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G} \).

a) \( J_{\text{min}} \) has \((\star) \Rightarrow J_{\text{min}} \) has \((\star\lambda)\).

b) \( J_{\text{max}} \) has \((\star) \Rightarrow J_{\text{max}} \) has \((\star\lambda)\).

**Example** We give an example of an infinitely generated \( G \in \mathcal{G}_{\text{dis}} \) such that

1. \( J_{\text{max}} \) does not have property \((\star\lambda)\),
2. \( \#\mathcal{J} = \aleph_0 \), and
there exists $J', J'' \in \mathcal{J}$ and $g \in G$ such that $J' < J''$ and $g^*(J') = g^*(J'')$.

Set $b_n = 2 - 1/n$ for $n \in \mathbb{N}$ and $\epsilon_n = \min\{\frac{b_{n+1}-b_n}{10}, \frac{b_n-b_{n-1}}{10}\}$. Choose polynomials $f_n$ and $g_n$ such that $J(f_n) = C(0, b_n)$ and $J(g_n) = C(\epsilon_n, b_n)$, where $C(z_0, r)$ denotes the circle of radius $r$ and center $z_0$. By choosing $m_n, j_n \in \mathbb{N}$ large enough, $h_n = f_n^{m_n}$ and $k_n = g_n^{j_n}$ will be such that $G = \langle h_n, k_n : n \in \mathbb{N} \rangle$ has the required properties. See [16] for details. In this example, $J_{\max} = C(0, 2)$ is contained in $\bigcup_{\lambda \in \Lambda} J(g_\lambda)$ which illustrates our next result, whose proof can be found in [16].

**Theorem 1.** Consider $G = \langle g_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}$ with $J(G)$ disconnected. Let $A = \bigcup_{\lambda \in \Lambda} J(g_\lambda)$ and denote by $M'$ and $M''$ the minimal and maximal (w.r.t. $\leq$) connected components of $\overline{A}$, respectively. Then both $J_{\min} \supset M'$ and $J_{\max} \supset M''$ and, in particular, both $J_{\min} \cap \overline{A} \neq \emptyset$ and $J_{\max} \cap \overline{A} \neq \emptyset$. Furthermore, we have the following.

1. If $J_{\min} \cap A = \emptyset$ (i.e., $J_{\min}$ does not have property $(\ast \lambda)$), then $J_{\min} = M'$ and $J_{\min}$ is the boundary of the unbounded component of $\mathbb{C} \setminus J_{\min}$.
2. If $J_{\max} \cap A = \emptyset$ (i.e., $J_{\max}$ does not have property $(\ast \lambda)$), then $J_{\max} = M''$ and $J_{\max}$ is the boundary of the bounded component of $\mathbb{C} \setminus J_{\max}$ which contains $J_{\min}$.

**Open Question 1:** If $J_{\max}$ fails to have $(\ast \lambda)$, then must $J_{\max}$ be a simple closed curve (or the common boundary of exactly two complementary domains)? Similarly, for $J_{\min}$.

**Open Question 2:** If $\# \mathcal{J} = \aleph_0$ and $G \in \mathcal{G}$ is finitely generated, then must every $J \in \mathcal{J}$ have property $(\ast)$?

**References**


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