THE INITIAL VALUE PROBLEMS IN CLIFFORD AND QUATERNION ANALYSIS

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ABSTRACT. Initial value problems of Cauchy-Kovalevskaya type can be solved in associated spaces provided the initial function belongs to this associated space (see [4]). The present paper will construct a method to solve this initial value problem.

1. INTRODUCTION

The initial value problem (I.V.P) of Cauchy-Kovalevskaya type has the form

\[
\begin{align*}
\frac{\partial w}{\partial t} &= L(t, x, w, \frac{\partial w}{\partial x_j}) & (1) \\
w(0, x) &= \varphi(x) & (2)
\end{align*}
\]

where \( w = w(t, x) \) is the founded function (or vector function) of time variable \( t \) and space variable \( x = (x_1, \ldots, x_n) \), \( L \) is the partial differential operator depending on \( t, x, w \) and \( \frac{\partial w}{\partial x_j} \) \( (j = 1, 2, \ldots, n) \) and \( \varphi \) is an initial given function. This problem can be solved by the contraction-mapping principle in case the initial function \( \varphi \) belongs to an associated space whose elements satisfy an interior estimate (see [6]).

In the present paper we consider the I.V.P (1) + (2) where the function \( w, \varphi \) and the partial differential operator \( L \) are taking value in a Clifford or Quaternion algebra.

2. PRELIMINARIES AND NOTATIONS.

Let \( e_1, e_2, \ldots, e_m \) be an orthonormal basis of the Euclidean space \( \mathbb{R}^n \) endowed with the standard Euclidean inner product \( \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j \). The Clifford algebra \( \mathcal{A} \) is then defined as the \( 2^m \)-dimensional real associative, but non-commutative algebra generated by \( \{e_1, \ldots, e_m\} \) and the multiplication rules: \( e_i e_j + e_j e_i = -2\delta_{ij} \). An element of \( \mathcal{A} \) is called a Clifford number and has the form \( a = \sum_{A \subseteq M} a_A e_A, a_A \in \mathbb{R} \), where \( A = (\alpha_1 \ldots \alpha_k) \), \( 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq m \) is a subset of \( M = \{1, 2, \ldots, m\} \) and \( e_A = e_{\alpha_1} \ldots e_{\alpha_k} \).

A function \( f \) defined in \( \Omega \subset \mathbb{R}^{n+1} \) taking value in Clifford algebra is a mapping \( f : \Omega \rightarrow \mathcal{A} \). Hence \( f \) has the form

\[
f = \sum_{A \subseteq M} f_A(x)e_A, \quad x = (x_0, x_1, \ldots, x_n).
\]

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If \( f_A \in C^k(\Omega) \) then we say that \( f \in C^k(\Omega, A) \).

Now, we define the Cauchy-Riemann operator as

\[
D := \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}
\]

and the Dirac operator as

\[
\sigma := \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}
\]

The definition of the monogenic functions is introduced as follow:

**Definition 1.** The function \( f \in C^2(\Omega, A) \) is called a monogenic function in \( \Omega \) if \( Df = 0 \) (or \( \sigma f = 0 \)).

By this definition we immediatly have a note

**Note:** If \( f \) is a monogenic function then its components \( f_A \) are harmonic functions.

In case \( m = 1 \), Clifford algebra \( A \) becomes the algebra \( \mathbb{C} \) of complex numbers, and the Cauchy-Riemann operator \( D \) is then \( \frac{\partial}{\partial \bar{z}} \).

In case \( m = 2 \), Clifford algebra \( A \) becomes the quaternion algebra \( \mathbb{H} \) and the Dirac operator has the form \( \sigma = i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \) and the Cauchy-Riemann operator has the form \( D = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \).

For other definitions concerning Cliffords and Quaternions we refer the reader to [1], [2].

### 3. The interior estimate for harmonic functions.

To solve this I.V.P by method contractive-mapping principle we need the property "interior estimate" of monogenic functions. But, first we construct the "interior estimate" for harmonic functions.

**Lemma 1.** Let \( u(x) \) be a harmonic function in \( \Omega \subset \mathbb{R}^n \) and continuous in \( \overline{\Omega} \). Then for each compact subset \( K \) of \( \Omega \) we have the estimate

\[
\|u(x)\|_K \leq \frac{n}{c.d} \|u\|_{\Omega}
\]

where \( d \) is the distance from \( K \) to \( \partial \Omega \); \( c \) is a constant depending on \( \Omega \); \( \|u\|_{\Omega} = \sup_{x \in \Omega} |u(x)| \).

(3) is called the "interior estimate" for harmonic function \( u \).

**Proof 1.** For any \( x^0 \in K \) we can choose a \( R > 0 \) so that the ball \( \{|x - x^0| = R\} \subset \Omega \) (\( R < \text{dist}(K, \partial \Omega) \)). From the Poisson-integral formula for harmonic functions it follows

\[
u(x) = \frac{1}{w_n R} \int_{|\xi - x^0| = R} u(\xi) \frac{R^2 - |x - x^0|^2}{|\xi - x|^n} d\mu
\]

where \( w_n \) is the eare of unit sphere in \( \mathbb{R}^n \).
Set \( r = |\xi - x| = \left[ \frac{1}{\sum_{j=1}^{n} (\xi_j - x_j)^2} \right]^{\frac{1}{2}} \).

We get
\[
\frac{\partial u}{\partial x_i} = \frac{1}{w_n R} \int_{|\xi-x^0|=R} u(\xi) \left[ \frac{-2(x_i - x_i^0)}{r^n} \right. \\
\left. - n(R^2 - |x - x^0|^2) \frac{x_i - \xi_i}{r^{n+2}} \right] d\mu.
\] (5)

Take \( r = R, x = x^0 \) we get
\[
\frac{\partial u}{\partial x_i}(x^0) = \frac{1}{w_n R} \int_{|\xi-x^0|=R} u(\xi) (-n) R^2 \frac{(x_i^0 - \xi_i)}{R^{n+2}} d\mu.
\] (6)

Since \( |x_i^0 - \xi_i| \leq |x^0 - \xi| = R \) it follows that
\[
\left| \frac{\partial u}{\partial x_i}(x^0) \right| \leq \frac{1}{w_n R} \int_{|\xi-x^0|=R} |u(\xi)| n \frac{1}{R^{n+1}} d\mu.
\] (7)

Hence
\[
\left| \frac{\partial u}{\partial x_i}(x^0) \right| \leq \frac{n}{R} \sup_{|\xi-x^0|=R} |u(\xi)|.
\] (8)

Because of the maximum principle for harmonic functions by using (8) we have
\[
\left| \frac{\partial u}{\partial x_i}(x^0) \right| \leq \frac{n}{R} \sup_{|\xi-x^0| \leq R} |u(\xi)|
\]
for each \( x^0 \in K \).

Then \( \| \frac{\partial u}{\partial x_i} \|_K \leq \frac{n}{R} \| u \|_\Omega \).

Let \( R \to d = \text{dist}(K, \partial \Omega) \) we get
\[
\left| \frac{\partial u}{\partial x_i} \right|_K \leq \frac{n}{d} \| u \|_\Omega.
\] (9)

\[
\Box
\]

4. Solving Initial Value Problem with the Monogenic Initial Function Taking Value in Clifford Algebra.

To solve this I.V.P we need the property "interior estimate" of the monogenic functions which is obtained by the fact that components of a monogenic function are harmonic functions. At first, we introduce an exhaustion of \( \Omega \subset \mathbb{R}^n \) by a family of subdomains \( \Omega_s \), \( 0 < s < s_0 \), satisfying the following conditions:

- To each point \( x \neq x_0 \) of \( \Omega \) where \( x_0 \in \Omega \) is fixedly chosen there exists uniquely determined \( s(x) \) with \( 0 < s(x) < s_0 \) such that \( x \in \partial \Omega_{s(x)} \)
- If \( 0 < s' < s'' < s_0 \), then \( \Omega_{s'} \) is a compact subset of \( \Omega_{s''} \).
• There exists a positive constant \( C_1 \) such that for any \( s', s'' \) with \( 0 < s' < s'' < s_0 \) the distance of \( \Omega_{s'} \) from \( \partial\Omega_{s''} \) can be estimated by:

\[
\text{dist}(\Omega_{s'}, \partial\Omega_{s''}) \geq C_1(s'' - s')
\]

where \( C_1 \) does not depend on the choice of \( s' \) and \( s'' \).

Define, finally \( s(x_0) = 0 \) then \( s_0 - s(x) \) is a measure of the distance of a point \( x \) of \( \Omega \) from the boundary \( \partial\Omega_{s(x)} \).

Now we consider the conical set

\[
M = \{(t, x) : x \in \Omega, 0 \leq t < \eta(s_0 - s(x))\}
\]

in the \( t, s \)-space. The parameter \( \eta \) will describe the height of the conical set and will be fixed later (theorem1). The base of \( M \) is the given domain \( \Omega \), whereas its lateral surface is defined by

\[
t = \eta(s_0 - s(x)).
\]

The nearer a point \( x \) to the boundary \( \partial\Omega \), the shorter the corresponding time interval (10). The expression

\[
d(t, x) = s_0 - s(x) - \frac{t}{\eta}
\]

is positive in \( M \), while it vanishes identically on the lateral surface of \( M \). Thus (11) can be interpreted as some pseudo-distance of a point \( (t, x) \) of \( M \) from the lateral surface of \( M \).

Again let \( \Omega_s, 0 < s < s_0 \), be an exhaustion of a given (bounded) domain in \( \mathbb{R}^n \). Let \( B_s \) be the space of all monogenic function in \( \Omega_s \) and \( f(x) \) equipped with the norm:

\[
\|f\|_{\Omega_s} = \text{Max}(\sup|f_A(x)|)
\]

Then the space \( B_s \) equipped with the norm \( \|\cdot\|_{\Omega_s} \) (which we denote by \( \|\cdot\|_s \) from now) turn out to be a Banach space.

For a fixedly chosen \( \tilde{t} < \eta s_0 \) the intersection of \( M \) with the plane \( t = \tilde{t} \) in the \( t, x \)-space is given by

\[
\{(t, x) : t = \tilde{t}; s(x) < \tilde{s}\} \quad \text{where} \quad \tilde{s} = s_0 - \frac{\tilde{t}}{\eta}.
\]

Let \( B_s(M) \) be the set of all function \( f = f(t, x) \) which are continuous in \( M \). We see \( f(\tilde{t}, x) \) belongs to \( B_s(s(x)) \) for fixed \( \tilde{t} \) if only \( s(x) < \tilde{s} \) where \( \tilde{s} \) is given by (12). And \( B_s(M) \) is equipped with the norm

\[
\|f\|_s = \sup_{(t, x) \in M} \|f(t, \cdot)\|_{s(x)} d(t, x) < +\infty.
\]

The definition (13) of the norm \( \|\cdot\|_s \) implies the estimate

\[
\|f(t, \cdot)\|_{s(x)} \leq \frac{\|f\|_s}{d(t, x)}
\]

for any point \( (t, x) \in M \).

**Proposition 1.** \( B_s(M) \) is a Banach space.
Proof 2. Note that the inequality \( d(t, x) \geq \delta > 0 \) defines a closed subset \( M_\delta \) of the conical domain \( M \). Each point of \( M \) is contained in such a subset \( M_\delta \) provided \( \delta \) is suitably choosen. For points \( (t, x) \) in \( M_\delta \), the definition (13) implies the estimate

\[
\|f(t, \cdot)\|_{s(x)} \leq \frac{1}{\delta} \|f\|_\star
\]

Now consider a fundamental sequence \( f_1, f_2, \ldots \) with respect to the norm \( \|\cdot\|_\star \). Then one has

\[
\|f_n(t, \cdot) - f_m(t, \cdot)\|_{s(x)} \leq \frac{1}{\delta} \epsilon
\]

for points in \( M_\delta \) provided \( n \) and \( m \) are sufficiently large. This implies also

\[
|f_n - f_m| \leq \frac{1}{\delta} \epsilon
\]

for points in \( M_\delta \). Consequently, a fundamental sequence converges uniformly in each \( M_\delta \), i.e., the \( f_n \) have a continuous limit function \( f_\star(t, x) \) in \( M \). Similarly, estimate (15) shows that for \( t = \tilde{t} \) and \( s(x) < \tilde{s} \) the limits function belongs to \( B_{s(x)} \) because of the completeness of this space. Carrying out the limiting process \( m \to \infty \) in the inequality

\[
\|f_n - f_m\|_\star < \epsilon,
\]

it follows, finally, \( \|f_n - f_\star\|_\star \leq \epsilon \) and, therefore, \( \|f_\star\|_\star \) is finite. \( \Box \)

Applying the estimate (3) of Lemma 1 for \( K = \Omega_{s'} \) and \( \Omega = \Omega_{s''} \) and by also the definitoin of the norm

\[
\|f\|_s = \text{Max}(\sup|f_A(x)|)
\]

we get

Proposition 2. If \( 0 < s' < s'' < s_0 \) and \( u \) is monogenic function in \( \Omega \) then

\[
\left\| \frac{\partial u}{\partial x_i} \right\|_{s'} \leq \frac{3}{\text{dist}(\Omega_{s'}, \Omega_{s''})} \|u\|_{s''}
\]

The second, we also need the concept "associated differential operators" : Let \( L \) be a first order differential operator depending on \( t, x, w \) and on the spacelike first order
derivatives: \( \frac{\partial w}{\partial x_j} \), while \( l \) is a differential operator with respect to the space variables \( x_j \) whose coefficients do not depend on the time \( t \). Then \( L \) is called "associated" to \( l \) if \( L \) transforms solutions of \( lw = 0 \) into solutions of the same equation for fixedly chosen \( t \), i.e., \( lw = 0 \Rightarrow l[w] = 0 \).

And the final, we have the main theorem:

**Theorem 1.** The initial value problem (1)+(2) is solvable by the method of constructive-mapping principle if initial function \( \varphi \) has "interior estimate" and coefficients of the partial differential operator \( L \) satisfying conditions under which \( L \) is "associated" to the Dirac operator (or the Cauchy-Riemann operator).

**Proof 3.** At first, we see that the I.V.P (1)+(2) can be rewrit as follow:

\[
    w(t, x) = \varphi(x) + \int_0^t L(\tau, x, w(\tau, x), \frac{\partial w(\tau, x)}{\partial x_j})d\tau
\]

(16)

Consequently, the solution of the I.V.P (1)+(2) is a fixed point of the operator

\[
    Fw(t, x) = \varphi(x) + \int_0^t L(\tau, x, w(\tau, x), \frac{\partial w(\tau, x)}{\partial x_j})d\tau
\]

(17)

Now we consider the space \( B^D(M) \) which is defined as follows:

\[
    B^D(M) = \{ w \in B_* (M) : Dw(t,.) = 0 \quad \text{for each fixed } t \}\n\]

We see that \( B^D_* (M) \) is closed subset of \( B_* (M) \) in view the Weierstrass' convergence theorem. On the other hand, the proposition 1 states that \( B_* (M) \) is a Banach space, so that \( B^D_* (M) \) is also Banach space.

Since the operator \( L \) is associated to the operator \( D \), so the operator \( F \) [see (17)] will map \( B^D_* (M) \) into itself.

Using the proposition 2 we have the estimate:

\[
    \|Fw - Fv\|_* \leq \eta.C\|w - v\|_*
\]

Where \( C \) depends on coefficients of the operator \( L \).

So that, if \( \eta \) is small enough \( (\eta < \frac{1}{C}) \) then \( F \) is a contractive mapping from Banach space \( B^D_* (M) \) into itself. Otherwise, the I.V.P (1)+(2) has unique solution in \( B^D_* (M) \).

\( \Box \)

**References**


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