MODULUS INEQUALITY FOR GRAFTING AND ITS APPLICATION

YOHEI KOMORI

ABSTRACT. We estimate the Teichmüller distance between a Riemann surface $X$ and $\text{gra}_{\theta, \gamma}(X)$ obtained from $X$ by $\theta$-grafting along a simple closed geodesic $\gamma$. As an application we estimate the Teichmüller distance between the conformal boundary and the convex core boundary of some regular $b$-groups of type $(1,1)$ one of which gives a counterexample of $K = 2$ conjecture.

1. MAIN RESULT

1.1. Moduli of cylinders. Fix a Riemann surface $X$. A cylinder $C$ in $X$ is a doubly connected region in $X$. $C$ is conformally isomorphic to the round annulus

$$A_r := \{ z \in \mathbb{C} \mid 1 < |z| < r \}$$

for some $r > 1$. We define the modulus of $C$ by

$$m(C) := \frac{\log r}{2\pi}$$

which is a conformal invariant of $C$. In practice modulus is a quasiconformal invariant. Let $f$ be a $K$-quasiconformal mapping from $X$ to a Riemann surface $Y$, then we have the following modulus inequality (c.f. [1])

$$\frac{1}{K} \leq \frac{m(f(C))}{m(C)} \leq K.$$ 

Another important modulus inequality is called the Grötzsch inequality (see [1]): Suppose a simple closed curve $\gamma$ in $C$ divides $C$ into 2 subcylinders $C_1$ and $C_2$. Then

$$m(C) \geq m(C_1) + m(C_2)$$

where the equality holds only when the image of $\gamma$ in the round annulus $A_r$ is a concentric round circle.

1.2. Jenkins-Strebel differentials and cylinders. A holomorphic quadratic differential $\varphi$ on a Riemann surface $X$ defines a local chart $\int \sqrt{\varphi}$ whose transition function has a form $z \mapsto \pm z + \text{const.}$. Then we can consider horizontal leaves, the inverse image of lines parallel to the real axis in $C$. For a simple closed geodesic $\gamma$ on $X$, the Jenkins-Strebel differential $\varphi$ is a holomorphic quadratic differential whose horizontal leaves are simple homotopic to $\gamma$.

Key words and phrases. convex hull, hyperbolic geometry, Kleinian group, Teichmüller space.
In particular, the set of all closed horizontal leaves of Jenkins-Strebel differential defines a cylinder in \( X \) homotopic to \( \gamma \), called the Jenkins-Strebel cylinder, and we denote it by \( C_X(\gamma) \). The Jenkins-Strebel cylinder has the following extremal property about its modulus (see [8]): the modulus of the Jenkins-Strebel cylinder with respect to \( \gamma \) is bigger than or equal to that of any cylinder in \( X \) homotopic to \( \gamma \).

1.3. **Grafting deformation.** The grafting deformation \( gr_{\theta, \gamma}(X) \) of \( X \) is constructed by cutting \( X \) along the hyperbolic geodesic \( \gamma \) and inserting a Euclidean cylinder of height \( \theta \) and circumference \( l \) with no twist, where \( l \) is the hyperbolic length of \( \gamma \) in \( X \).

More precisely consider the upper half plane \( \mathbb{H}^2 \) as the universal covering space of \( X \) so that the imaginary axis is a lift of \( \gamma \) in \( \mathbb{H}^2 \). Then \( g(z) = e^{i\theta} z \) is a generator of the stabilizer subgroup of the covering transformation group \( \Gamma \). Next take the sector domain \( \{ z \in \mathbb{C} \mid 0 < \arg(z) < \theta \} \) with the metric \( |dz|/|z| \) on which \( g \) also acts and the quotient space is a Euclidean cylinder of height \( \theta \) and circumference \( l \). Cut \( \mathbb{H}^2 \) along the imaginary axis and insert this sector. After doing the same operations on \( \mathbb{H}^2 \) along all lifts of \( \gamma \), we have a new simply connected Riemann surface on which \( \Gamma \) also acts and the quotient space is \( gr_{\theta, \gamma}(X) \).

1.4. **Main result.**

**Theorem 1.** Suppose that a simple closed geodesic \( \gamma \) in \( X \) divides the Jenkins-Strebel cylinder \( C_X(\gamma) \) into two subcylinders \( C_1 \) and \( C_2 \). Then the maximal dilatation \( K \) of the Teichmüller mapping between \( X \) and \( gr_{\theta, \gamma}(X) \) satisfies

\[
K \geq \frac{1}{m} \left( m_1 + m_2 + \frac{\theta}{l} \right)
\]

where \( l \) is the hyperbolic length of \( \gamma \) and \( m, m_1 \) and \( m_2 \) are moduli of \( C_X(\gamma), C_1 \) and \( C_2 \) respectively. In particular if the geodesic \( \gamma \) is also a closed leaf of \( C_X(\gamma) \) then

\[
K \geq 1 + \frac{\theta}{ml}.
\]

**Proof.** Let \( f \) be the Teichmüller mapping from \( Y = gr_{\theta, \gamma}(X) \) to \( X \). Then the quasi-conformal invariance of the modulus and the extremal property of the Jenkins-Strebel cylinder on modulus implies

\[
\frac{1}{K} \leq \frac{m(f(C_Y(\gamma)))}{m(C_Y(\gamma))} \leq \frac{m(C_X(\gamma))}{m(C_Y(\gamma))} = \frac{m}{m(C_Y(\gamma))}.
\]

Since \( Y \) is obtained by \( \theta \)-grafting along \( \gamma \), \( Y \) contains the cylinder \( C \) consisting of \( C_1, C_2 \) and the inserted cylinder whose modulus is \( \theta/l \). Then Grötzsch inequality implies that the modulus of \( C \) is bigger than or equal to \( m_1 + m_2 + \theta/l \) and equality holds only if \( \gamma \) is a closed leaf of \( C_X(\gamma) \). The extremal property of Jenkins-Strebel cylinder on modulus implies

\[
m(C_Y(\gamma)) \geq m(C) = m_1 + m_2 + \theta/l.
\]

Inequalities (1) and (2) conclude the assertion. \( \Box \)
2. Application

Let $G$ be a Kleinian group, a discrete subgroup of $\text{PSL}_2(\mathbb{C})$ acting on the Riemann sphere $\hat{\mathbb{C}}$ conformally by linear fractional transformations. We assume that $G$ has a simply connected invariant component $\Omega$ of the region of discontinuity. Then the quotient space $\Omega/G$ has a complex structure induced by $\Omega \subset \hat{\mathbb{C}}$.

Thinking of $\hat{\mathbb{C}}$ as the boundary of hyperbolic 3-space $\mathbb{H}^3$, the hyperbolic convex hull of $\hat{\mathbb{C}} - \Omega$ is the smallest closed set in $\mathbb{H}^3$ which contains every hyperbolic geodesic arc with endpoints in $\hat{\mathbb{C}} - \Omega$. The boundary of the hyperbolic convex hull of $\hat{\mathbb{C}} - \Omega$ in $\mathbb{H}^3$ is called the dome of $\Omega$ and is denoted by $\text{Dome}(\Omega)$. Then $G$ also acts on $\mathbb{H}^3$ as a group of hyperbolic isometries by means of the Poincaré extension. Since $\Omega$ is an invariant component of $G$, $G$ also preserves $\text{Dome}(\Omega)$ and the quotient space $\text{Dome}(\Omega)/G$ has a hyperbolic structure. As $\Omega/G$ and $\text{Dome}(\Omega)/G$ have the same topological type, they define two points in the Teichmüller space $\mathcal{T}$.

As an application of Theorem 1, we will estimate the Teichmüller distance between $\Omega/G$ and $\text{Dome}(\Omega)/G$ from below when $G$ is the following three cases; $G$ is generated by $S$ and $T_\mu$

$$S(z) = z + 2, \quad T_\mu(z) = \frac{1}{z} + \mu$$

where $\mu = 1 + 2i, 1 + \sqrt{5}i$ and $1 + \sqrt{7}i$.

2.1. Models of Teichmüller space of once-punctured tori. Any (marked) once-punctured torus is compactified by a (marked) flat torus $\mathbb{C}/(1 \cdot Z + \tau \cdot Z)$ where $\tau \in \mathbb{H}^2$. Therefore $\mathbb{H}^2$ can be considered as Teichmüller space of once-punctured tori. On the other hand (marked) once-punctured torus is uniformized by a (marked) fuchsian once-punctured torus group $\Gamma = \langle A, B \rangle$ which is unique up to conjugation in $\text{PSL}(2, \mathbb{R})$; such a torus is also uniquely determined by the triple $(x, y, z) = (\text{Tr} A, \text{Tr} B, \text{Tr} AB) \in \mathbb{R}^3$. Hence Teichmüller space of once-punctured tori is also realized as a semi-algebraic subset of the real algebraic set defined by the equation $x^2 + y^2 + z^2 = xyz$.

2.2. The action of the modular group $\text{PSL}(2, \mathbb{Z})$. The modular group $\text{PSL}(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{H}^2$ and on the trace parameter space $(\text{Tr} A, \text{Tr} B, \text{Tr} AB) \in \mathbb{R}^3$ as models of the Teichmüller space of once-punctured tori (see [2]). The modular group is generated by the elements $J = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and $N = \left( \begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The transformation $J$ changes the marking $\langle A, B \rangle$ to $\langle B^{-1}, A \rangle$, and thus changes the trace parameters $(x, y, z)$ to $(y, x, xy - z)$. The transformation $N$ changes the marking $\langle A, B \rangle$ to $\langle B^{-1}, AB \rangle$, and thus changes the trace parameters $(x, y, z)$ to $(y, z, x)$. We note here that the element $M = \left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ changes the marking $\langle A, B \rangle$ to $\langle AB, B \rangle$ and the trace parameters $(x, y, z)$ to $(z, y, yz - x)$. The following result is well-known (see [2] and [4]):

**Proposition 1.** Let $\Gamma = \langle A, B \rangle$ be a fuchsian once-punctured torus group. Then the Teichmüller parameter of $\Gamma$ in $\mathbb{H}^2$ is equal to

$$\omega = -1/2 + \sqrt{3}i/6, \quad -1/2 + i/2 \quad \text{and} \quad -1/2 + \sqrt{3}i/2$$

if and only if

$$(\text{Tr} A, \text{Tr} B, \text{Tr} AB) = (6, 3, 3), \quad (4, 2\sqrt{2}, 2\sqrt{2}) \quad \text{and} \quad (3, 3, 3)$$

respectively.
2.3. Fenchel-Nielsen coordinates. Next we consider the Fenchel-Nielsen coordinates of the Teichmüller space of once-punctured tori (see [6]). Let X be a once-punctured torus uniformized by $\Gamma = \langle A, B \rangle$, and $\gamma$ be a simple closed geodesic on X representing $A \in \Gamma$. We denote its hyperbolic length by $l$. Then $X \setminus \gamma$ is a once-punctured cylinder with geodesic boundaries having the same length $l$. It is uniformized by $\langle A, A' \rangle$ where

$$
A = \begin{pmatrix} \cosh l/2 & \cosh l/2 + 1 \\ \cosh l/2 - 1 & \cosh l/2 \end{pmatrix},
$$

$$
A' = \begin{pmatrix} \cosh l/2 & \cosh l/2 - 1 \\ \cosh l/2 + 1 & \cosh l/2 \end{pmatrix}.
$$

The original surface X can be reconstructed by gluing together the geodesic boundaries of $X \setminus \gamma$. In terms of Fuchsian groups, this operation can be realized by forming the HNN extension of $\langle A, A' \rangle$ by an element $B \in \text{PSL}(2, \mathbb{R})$ satisfying

$$
B^{-1}AB = A'.
$$

From this condition $B$ can be written as

$$
B = \begin{pmatrix} \cosh \tau/2 \coth l/4 & -\sinh \tau/2 \\ -\sinh \tau/2 & \cosh \tau/2 \tanh l/4 \end{pmatrix},
$$

where $\tau$ is a free real parameter which has the following geometric interpretation: if the common perpendicular $\delta$ to the hyperbolic axis of A and that of $B^{-1}AB = A'$ meets these axes in points X and Y, then $\tau$ is the signed distance from X to $B(Y)$ where the axis of A is oriented from the attracting fixed point $\cosh l/4$, to the repelling fixed point $-\coth l/4$. The map $\Gamma \mapsto (l, \tau)$ gives the Fenchel-Nielsen coordinates of the Teichmüller space of once-punctured tori and $l$ and $\tau$ are called the length and twist parameters respectively. Let us compute the Fenchel-Nielsen coordinates of the marked torus whose Teichmüller parameter is $-1/2 + i \in \mathbb{H}^2$.

**Proposition 2.** For a once-punctured torus whose Teichmüller parameter is $-1/2 + i \in \mathbb{H}^2$, the twist parameter is the half of its length parameter. In particular the Fenchel-Nielsen coordinates of the once-punctured torus whose Teichmüller parameter is

$$
\omega = -1/2 + \sqrt{3}i/6, \ -1/2 + i/2 \ \text{and} \ -1/2 + \sqrt{3}i/2
$$

are given by

$$(l, \tau) = (2 \cosh^{-1} 3, \cosh^{-1} 3), \ (2 \cosh^{-1} 2, \cosh^{-1} 2), \ \text{and} \ (2 \cosh^{-1} 3/2, \cosh^{-1} 3/2)$$

respectively.

2.4. Pure bending deformation. Let $\Gamma_0 = \langle A_0, B_0 \rangle$ be the Fuchsian group whose quotient surface $(\mathbb{H}^2/\Gamma_0; A, B_0)$ has Teichmüller parameter of the form $-1/2 + i \in \mathbb{H}^2$, and let $l_0$ and $\tau_0 = l_0/2$ be the Fenchel-Nielsen coordinates of it stated in Proposition 2. Now we start to deform $\Gamma_0$ in $\text{PSL}(2, \mathbb{C})$ to get the Kleinian group we are looking for. Keeping $l_0$ fixed, let us release the twist parameter $\tau$ as a free complex parameter. Then we can still consider the subgroup $\Gamma(\tau) = \langle A, B \rangle$ of $\text{PSL}(2, \mathbb{C})$ defined by equations (3) and (5) which acts on $\mathbb{H}^3$ as a group of hyperbolic isometries by means of the Poincaré extension. In particular there is a natural homomorphism $f_\tau$ from $\Gamma_0 = \Gamma(\tau_0)$ to $\Gamma(\tau)$ defined by $f_\tau(A_0) = A$ and $f_\tau(B_0) = B$. Here we should remark that $A = A_0$ and $f_\tau(A') = A'$ whereas $f_\tau(B_0) \neq B_0$ in general. Hence $\Gamma_0$ and $\Gamma(\tau)$ share the same subgroup $\langle A, A' \rangle$ on which $f_\tau$ is identity.
Associated with \( \tau \in \mathbf{C} \), we can consider the map \( \psi_\tau : \mathbf{H}^2 \to \mathbf{H}^3 \), called the pleated surface for \( f_\tau \), defined as follows (see [3]): Consider \( \mathbf{H}^2 \) as the totally geodesic surface in \( \mathbf{H}^3 \) defined by \( \text{Im} \ z = 0 \), and remove the \( \Gamma_0 \)-orbit of the axis of \( A_0 \) from \( \mathbf{H}^2 \). Take the connected component whose boundary contains the axis of \( A \) and that of \( A' \), and consider its closure in \( \mathbf{H}^3 \). We call it the flat piece and denote it by \( F \). By means of this flat piece \( F \), we can define the pleated surface \( \psi_\tau \) as follows:

\[
\psi_\tau(z) = \begin{cases} 
z & \text{if } z \in F \\ f_\tau(g)g^{-1}(z) & \text{if } z \in g(F) \text{ for } g \in \Gamma_0. \end{cases}
\]

We remark that the stabilizer of \( F \) in \( \Gamma_0 \) is \( \langle A, A' \rangle \), the subgroup on which \( f_\tau \) is the identity, hence, \( \psi_\tau \) is well-defined. The pleated surface \( \psi_\tau \) satisfies the \( f_\tau \)-equivariant property

\[
\psi_\tau(g(z)) = f_\tau(g)\psi_\tau(z)
\]

for \( z \in \mathbf{H}^2 \) and \( g \in \Gamma_0 \). We denote \( \psi_\tau(\mathbf{H}^2) \) by \( \mathbf{H}^2(\tau) \) which is the union of the \( \Gamma(\tau) \)-orbit of \( F \). In particular when \( \tau = \tau_0 + i\theta \), adjacent flat pieces \( g_1(F) \) and \( g_2(F) \) of \( \mathbf{H}^2(\tau) \) are bent along the axis of some conjugate of \( A \) in \( \Gamma(\tau) \) with bending angle \( \theta \). This process is called pure bending. We denote \( \Gamma(\tau_0 + i\theta) \) and \( \mathbf{H}^2(\tau_0 + i\theta) \) by \( \Gamma(\theta) \) and \( \mathbf{H}^2(\theta) \). Assuming that \( |\theta| \) is sufficiently small, it is shown in [3] that \( \Gamma(\theta) \) is a quasi-Fuchsian group and \( \mathbf{H}^2(\theta) \) is \( \text{Dome}(\Omega) \) of the invariant component \( \Omega \) of \( \Gamma(\theta) \). From construction \( (\text{Dome}(\Omega)/\Gamma(\theta); A, B) \) is conformally isomorphic to \( (\mathbf{H}^2/\Gamma_0; A, B_0) \) as a marked surface. About pure bending, we have (see [4])

**Proposition 3.** Define the real number \( \theta_0 \) by \( -\pi < \theta_0 < 0 \) and

\[
\cos \theta_0 = 1 - \frac{1}{\cosh \frac{l}{2}} - \frac{1}{\cosh^2 \frac{l}{2}}.
\]

(1) \( AB^2 \in \Gamma(\theta) \) is purely hyperbolic for \( \theta_0 < \theta < 0 \) while it is parabolic when \( \theta = \theta_0 \).

(2) For \( \theta_0 < \theta < 0 \), \( \Gamma(\theta) \) is a quasi-Fuchsian punctured torus group whose convex core has two boundary components which are bent along simple closed geodesics represented by \( A \) and \( AB^2 \) respectively.

(3) \( \Gamma(\theta_0) \) is a regular \( b \)-group of type \( (1, 1) \) in which \( AB^2 \) is accidentally parabolic.

From the previous bending construction, the marked surface \( (\text{Dome}(\Omega)/\Gamma(\theta_0); AB^2, B) \) is conformal to \( (\mathbf{H}^2/\Gamma_0; AB^2, B) \). Hence considering the action of the modular transformation \( M(\tau) = \frac{\tau}{\tau+1} \in \text{PSL}(2, \mathbf{Z}) \), the Teichmüller parameter of \( (\mathbf{H}^2/\Gamma_0; AB^2, B) \) is \( M^2(\omega) = \frac{1}{2} + \frac{i}{4t} \). Now \( \Gamma(\theta_0) \) is \( \text{PSL}(2, \mathbf{C}) \) conjugate to the group \( G \) generated by \( S \) and \( T \)

\[
S(z) = z + 2, \quad T(z) = \frac{1}{z} + 1 + \frac{i}{2t}
\]

and the Teichmüller parameter of \( (\text{Dome}(\Omega)/G; S, T) \) is \( \frac{1}{2} + \frac{i}{4t} \).

**Corollary 1.** For the group \( G \) generated by \( S \) and \( T_\mu \)

\[
S(z) = z + 2, \quad T_\mu(z) = \frac{1}{z} + \mu.
\]

where

\[
\mu = 1 + \sqrt{7}i, \quad 1 + \sqrt{5}i \quad \text{and} \quad 1 + 2i,
\]
the Teichmüller parameter of \((\text{Dome}(\Omega)/G; S, T)\) is
\[
\omega = 1/2 + (\sqrt{3}/2)i, \ 1/2 + (1/2)i \quad \text{and} \quad 1/2 + (\sqrt{3}/6)i.
\]

2.5. Result. Since the boundary of \(C_\chi(\gamma)\) is a hyperbolic geodesic from the puncture to itself when \(X\) has a half Dehn twist (see Lemma 1.4 and Figure 4 of [5]), \(\gamma\) divides \(C_\chi(\gamma)\) into two subcylinders. We summarize our computation in Table 1 below. For the groups stated in Corollary 1, \(m, l\) and \(\theta\) can be calculated by Proposition 1, 2 and the equation (6) respectively. To apply Theorem 1, all we need is to compute the moduli of subcylinders, \(m_1\) and \(m_2\) only where we need a computer to get approximation values of them: Robert Silhol wrote a very effective Maple program which gives us the modulus of the hyperbolic quadrilateral with angle 0, \(\pi/2, \pi/2, \pi/2\) ([7]). As a result we can estimate the Teichmüller distance \(1/2 \log K\) between \(\Omega/G\) and \(\text{Dome}(\Omega)/G\) from below. In particular the last line of Table 1 gives a counterexample to the equivariant \(K=2\) conjecture (see [4]).

<table>
<thead>
<tr>
<th>(\mu)</th>
<th>1</th>
<th>(\theta)</th>
<th>(m)</th>
<th>(m_1 = m_2)</th>
<th>(K &gt;)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 + \sqrt{7}i)</td>
<td>(2 \cosh^{-1} 3)</td>
<td>0.981765</td>
<td>(\sqrt{3}/6)</td>
<td>0.2726372</td>
<td>1.90911</td>
</tr>
<tr>
<td>(1 + \sqrt{5}i)</td>
<td>(2 \cosh^{-1} 2)</td>
<td>1.31812</td>
<td>(1/2)</td>
<td>0.4949395</td>
<td>1.99069</td>
</tr>
<tr>
<td>(1 + 2i)</td>
<td>(2 \cosh^{-1} 3/2)</td>
<td>1.68214</td>
<td>(\sqrt{3}/2)</td>
<td>0.8654479</td>
<td>2.00843</td>
</tr>
</tbody>
</table>

Table 1

REFERENCES


(Yohei Komori) DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN

E-mail address: komori@sci.osaka-cu.ac.jp