SOME CONDITIONS FOR SEMI-HYPERBOLICITY OF ENTIRE FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We give a necessary and sufficient condition for the semi-hyperbolicity of an entire function \( f \) at a point in the Julia set. We also show some results on measure theoretical properties of the dynamics of \( f \) as an application.

1. Semi-hyperbolicity

Let \( f \) be an entire function, \( J(f) \) be its Julia set and \( F(f) \) be its Fatou set. The following is a part of the Mañé's theorem for rational functions:

**Theorem 1** (Mañé, [Ma]). Let \( f \) be a rational function and \( z_0 \in J(f) \). Suppose that

(i) \( z_0 \) is not a parabolic periodic point and (ii) \( z_0 \notin \bigcup_{c \in \text{Rec} \cap J(f)} \omega(c) \),

where \( \omega(c) \) is the \( \omega \)-limit set of \( c \) and

\[
\text{Rec} = \{ c \mid c \text{ is a recurrent (i.e. } c \in \omega(c) \text{) critical point of } f \}\.
\]

Then for every \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( z_0 \) which satisfies the following:

(1) For every \( n \in \mathbb{N} \) and every connected component \( V \) of \( f^{-n}(U) \),

\[
\text{diam}_{\text{sph}}(V) \leq \varepsilon
\]

holds, where \( \text{diam}_{\text{sph}} \) denotes the spherical diameter on \( \hat{\mathbb{C}} \).

(2) There exists an \( N \in \mathbb{N} \) such that for any connected component \( V \) of \( f^{-n}(U) \) (\( \forall n \)), \( f^n|_V : V \to U \) satisfies

\[
\deg(f^n|_V : V \to U) \leq N.
\]

See also [CJY]. Taking this result into account, we define the semi-hyperbolicity of \( f \) at a point \( z_0 \in J(f) \) as follows:

**Definition 1.** \( f \) is semi-hyperbolic at \( z_0 \in J(f) \) if there exists a neighborhood \( U \) of \( z_0 \) such that the condition (2) in Theorem 1 holds. In the case that \( f \) is transcendental, we add the following property: \( f^n|_V : V \to U \) is proper for every \( V \). Recall that \( f : X \to Y \) is called proper if \( f^{-1}(K) \subset X \) is compact for every compact subset \( K \subset Y \). Note that this property is automatically satisfied when \( f \) is a polynomial or rational. We say \( f \) is semi-hyperbolic if \( f \) is semi-hyperbolic at any point \( z_0 \in J(f) \).

The converse of Theorem 1 is also true. That is, if \( z_0 \) is a parabolic periodic point or \( z_0 \in \bigcup_{c \in \text{Rec} \cap J(f)} \omega(c) \), then \( f \) is not semi-hyperbolic at \( z_0 \in J(f) \). In this paper we investigate a condition for semi-hyperbolicity for transcendental entire functions. In

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transcendental case, a new phenomena can occur. For example, Bergweiler and Morosawa ([BM]) constructed an example of \( f \) with no parabolic periodic point and no recurrent critical point, but has a point \( z_0 \in J(f) \) at which \( f \) is not semi-hyperbolic.

2. CHARACTERIZATION OF SEMI-HYPERBOLICITY FOR ENTIRE FUNCTIONS

Define the sets Rec, Non-Rec and AV as follows:

\[
\begin{align*}
\text{Rec} &:= \{c \mid c \text{ is a recurrent (i.e. } c \in \omega(c)\text{) critical point of } f\}, \\
\text{Non-Rec} &:= \{c \mid c \text{ is a non-recurrent (i.e. } c \notin \omega(c)\text{) critical point of } f\}, \\
\text{AV} &:= \{c \mid c \text{ is an asymptotic value of } f\}.
\end{align*}
\]

For a critical point \( c \), let \( \text{ord}(c) \) denote the order of \( c \). Then the main result of this paper is the following (For the details of the proof, see [K1]):

**Theorem A.** Let \( f \) be a (transcendental) entire function and \( z_0 \in J(f) \). Then \( f \) is semi-hyperbolic at \( z_0 \) if and only if \( z_0 \notin Z \), where the set \( Z \) is defined as follows:

\[
Z = \left( \bigcup_{i=1}^{3} X_i \right) \cup \left( \bigcup_{j=1}^{5} Y_j \right),
\]

where

\[
X_1 = \{p \mid p \text{ is a parabolic periodic point of } f\},
\]

\[
X_2 = \{p \mid p \text{ is an attracting periodic point of } f\},
\]

\[
X_3 = \{p \mid f^n|_W \to p (n_i \to \infty) \text{ for some wandering domain } W\},
\]

\[
Y_1 = \bigcup_{c \in \text{Rec} \cap J(f)} \omega(c), \quad Y_2 = \bigcup_{n=0}^{\infty} f^n(\text{AV}) \cap J(f),
\]

\[
Y_3 = \{p \mid \forall U : \text{nbhd of } p, \quad \forall n \in \mathbb{N}, \quad \forall \varepsilon > 0, \quad \exists V : \text{open set with } \text{diam}(V) < \varepsilon \text{ s.t. } V \text{ contains more than } n \text{ crit. pts (counting multiplicity),} \quad \exists m, \quad f^m(V) \subset U \text{ and } \text{diam}(f^i(V)) < \varepsilon (1 \leq i \leq m)\},
\]

\[
Y_4 = \left\{ p \mid p = \lim_{i \to \infty} f^n(c_i), \quad c_i \in \text{Non-Rec} \cap J(f) \text{ (} i \in \mathbb{N} \text{) are mutually different with } \sup_i \text{ord}(c_i) < \infty \text{ and for any } \varepsilon > 0 \text{ let} \\
N_i(\varepsilon) := \#\{c \mid c : \text{critical point, } O^+(f(c_i)) \cap U_\varepsilon(c) \neq \emptyset\} \text{ then } \sup_i N_i(\varepsilon) = \infty \right\},
\]

\[
Y_5 = \left\{ p \mid p = \lim_{i \to \infty} f^n(c_i), \quad c_i \in \text{Non-Rec} \cap J(f) \text{ (} i \in \mathbb{N} \text{) are mutually different with } \sup_i \text{ord}(c_i) < \infty \text{ and let} \\
\delta_i(n) := \sup \{\delta \mid \#\{O^+(f(c_i)) \cap (U_\delta(c_i)) \leq n\} \} \text{ then } \inf_i \delta_i(n) = 0 \text{ for } \forall n \right\}.
\]

(Outline of the proof): Suppose \( z_0 \in J(f) \) and \( z_0 \notin Z \), then take a neighborhood \( U \) of \( z_0 \) with \( \overline{U} \cap Z = \emptyset \). For \( z \in U \) let \( S(z, \varepsilon) \) be a square centered at \( z \) with side length \( 2\varepsilon \).
and with sides parallel to coordinate axes. We say $S(z, \varepsilon)$ is admissible if $S(z, 3\varepsilon) \subset U$.

First we show the following:

Lemma 1. For a given $\varepsilon > 0$ and an $N \in \mathbb{N}$, there exists a $\delta > 0$ which satisfies the following: If $S(z, \delta)$ is an admissible square and $S_n$ is a connected component of $f^{-n}(S(z, \delta))$ such that $\deg(f^n|_{S_n}) \leq N$, then

$$\text{diam}(f^{-n}(S(z, \delta/2))) \leq \varepsilon$$

holds for the same branch of $f^{-n}$.

Now since $z_0 \notin Z$, there is a neighborhood $U$ of $z_0$ satisfying

(0) $U$ does not contain attracting periodic points, parabolic periodic points, wandering domains, points in orbits of recurrent critical points or asymptotic values.

Moreover, $U$ satisfies either one of the following:

(1) The number of critical points with $O^+(f(c)) \cap U \neq \emptyset$ is finite (let us denote them by $c_1, c_2, \cdots, c_{n_0}$) and all of them are non-recurrent. Then for some $\varepsilon_0 > 0$ we have

$$O^+(f(c_i)) \cap U_{\varepsilon_0}(c_i) = \emptyset.$$

(2) The number of critical points with $O^+(f(c)) \cap U \neq \emptyset$ is infinite (let us denote them by $c_1, c_2, \cdots$) and all of them are non-recurrent. There exists an $M_0 > 0$ such that

$$\text{ord}(c_i) \leq M_0, \text{ for } \forall i \in \mathbb{N}.$$

Also there exists an $\varepsilon_1 > 0$ and an $N_0 \in \mathbb{N}$ such that

$$\# \{ c \mid c: \text{critical point, } O^+(f(c_i)) \cap U_{\varepsilon_1}(c_i) \neq \emptyset \} \leq N_0 < \infty$$

holds for every $i \in \mathbb{N}$. Furthermore there exists a $\delta_1 > 0$ and an $n_1 \in \mathbb{N}$ such that

$$\# \{ O^+(f(c_i)) \cap U_{\delta_1}(c_i) \} \leq n_1, \text{ for } \forall i \in \mathbb{N}.$$ 

In this case, we put $\varepsilon_0 := \min(\varepsilon_1, \delta_1)$.

Now let $N := (M_0 + 1)^{N_0(n_1 + 1)}$ and take $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/36N$ (In the case of (1), we take $M_0 := \max_{1 \leq i \leq n_0} \text{ord}(c_i)$, $N_0$ similar above and $n_1 := 1$). Then there is a $\delta > 0$ which is determined by the previous Lemma 1.

Lemma 2. For any $\eta$ with $0 < \eta \leq \delta$ and $n \in \mathbb{N}$, we have

$$\text{diam}(f^{-n}(S(z_0, \eta/2))) \leq \varepsilon.$$

That is, the conclusion of Lemma 1 holds without the assumption on degree.

Hence for any $\varepsilon > 0$ with $\varepsilon < \varepsilon_0/36N$ by taking $\sigma > 0$ sufficiently small, we have

$$\text{diam}(f^{-n}(S(z_0, \sigma))) \leq \varepsilon, \text{ for } \forall n.$$

With a little more argument, we can conclude

$$\deg(f^n|_{S(z_0, \sigma)}) < N = (M_0 + 1)^{N_0(n_1 + 1)}.$$

It is rather easy to check that $z_0 \in Z$ implies that $f$ is not semi-hyperbolic at $z_0$.

3. Applications

As an application of Theorem A, we can show the following result on a measure theoretical property for the dynamics of entire functions (For the details of the proof, see [K2]). This is a refinement of the result by Bock ([B]).
**Theorem B** Either one of the following (AT$\tilde{Z}$) or (ERG) holds for the dynamics of an entire function $f$:

$$(\text{AT$\tilde{Z}$})$$
Almost every point $z \in J(f)$ is attracted to the set $\tilde{Z}$, that is,
$$\lim_{n \to \infty} \text{dist}_{	ext{sph}}(f^n(z), \tilde{Z}) = 0, \quad (\text{i.e. } \omega(z) \subseteq \tilde{Z})$$
holds for a.e. $z \in J(f)$, where $\tilde{Z} := Z \cup \{\infty\}$ and $Z$ is the set in Theorem A.

$$(\text{ERG})$$
$J(f) = \mathbb{C}$ and $f$ is ergodic.

Furthermore, (ERG) can be replaced by the following (IR) or (FOD):

$$(\text{IR})$$
$J(f) = \mathbb{C}$ and $f$ is infinitely recurrent, i.e. for every $X \subset \mathbb{C}$ with $\text{Leb}(X) > 0$ and every $z \in \mathbb{C},$
$$\#\{n \in \mathbb{N} \mid f^n(z) \in X\} = \infty$$
holds, where $\text{Leb}(\cdot)$ denotes the Lebesgue measure on $\mathbb{C}$.

$$(\text{FOD})$$
$J(f) = \mathbb{C}$ and for a.e. $z \in \mathbb{C}$, the forward orbit $O^+(z) \subset \mathbb{C}$ is dense.

**(Outline of the proof):** For the proof, we will use the next Lemma:

**Lemma 3.** Define the set $B$ as follows:
$$B := \{z \in J(f) \mid \limsup_{n \to \infty} \text{dist}_{	ext{sph}}(f^n(z), \tilde{Z}) > 0\}.$$ 

If $X \subseteq J(f)$ satisfies $f(X) \subseteq X$ and $\text{Leb}(X \cap B) > 0$, then $\hat{\text{Leb}}(X) = 1$, that is, $X$ has full measure in $\mathbb{C}$, where $\hat{\text{Leb}}(\cdot)$ denotes the normalized Lebesgue measure on $\hat{\mathbb{C}}$. □

Now suppose (AT$\tilde{Z}$) does not hold, then we have $\hat{\text{Leb}}(B) > 0$. Then by applying Lemma 3 with $X = B$, we conclude that $\hat{\text{Leb}}(B) = 1$ and in particular $\hat{\text{Leb}}(J(f)) = 1$, hence $J(f) = \mathbb{C}$. Now suppose that $f^{-1}(A) = A$ and $\hat{\text{Leb}}(A) > 0$, then it follows that $\hat{\text{Leb}}(A) = 1$ from Lemma 3 again. Thus (ERG) holds.

Next we show that (IR) holds when (AT$\tilde{Z}$) does not hold. Suppose not, then there exists a set $A$ with $\hat{\text{Leb}}(A) > 0$ such that $\hat{\text{Leb}}(\{z \mid \#\{n \mid f^n(z) \in A\} < \infty\}) > 0$. Then for some $n_0 \in \mathbb{N}$ we have $\hat{\text{Leb}}(Y) > 0$, where
$$Y = \{z \mid f^n(z) \notin A \text{ for } \forall n \geq n_0\}.$$ 

The set $Y$ satisfies $f(Y) \subseteq Y$. Also since $\hat{\text{Leb}}(B) = 1$, we have
$$\hat{\text{Leb}}(Y \cap B) = \hat{\text{Leb}}(Y) > 0.$$ 

Hence it follows that $\hat{\text{Leb}}(Y) = 1$ from Lemma 3. On the other hand, since $f^{-n_0}(A) \cap Y = \emptyset$, we have $\hat{\text{Leb}}(f^{-n_0}(A)) = 0$. Thus $\hat{\text{Leb}}(A) = 0$, which is a contradiction.

The proof of (FOD) is straightforward. □

**Corollary C** Let $f$ be an entire function with

(i) Every critical point $c$ is either preperiodic or satisfies $f^n(c) \to \infty$ ($n \to \infty$),

(ii) Every asymptotic value is eventually periodic,

(iii) The post-singular set $P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))$ is discrete in $\mathbb{C}$.

Then either one of the following holds:
(MGA) \{\infty\} is a metric global attractor, that is,
\[ f^n(z) \to \infty \ (n \to \infty) \text{ for a.e. } z \in \mathbb{C} \ (\text{i.e. } \omega(z) = \{\infty\}). \]

(FOD) \(J(f) = \mathbb{C}\) and \(O^+(z) \subset \mathbb{C}\) is dense for a.e. \(z \in \mathbb{C}\) (i.e. \(\omega(z) = \mathbb{C}\)).

In particular, if \(f\) satisfies the conditions (i) \(\sim\) (iii) and \(J(f) \neq \mathbb{C}\), then \(\{\infty\}\) is a metric global attractor for \(f\).

(Outline of the proof): It follows from (i) \(\sim\) (iii) that every singular value \(p\) satisfies either \(f^n(p) \to \infty\) or eventually lands on a repelling periodic point. If \(F(f) \neq \emptyset\), then only possible Fatou components are either Baker domains (or their preimages) or wandering domains. If there is a wandering domain \(U\), then we have \(f^n|_U \to \infty\), because in general a finite limit function on a wandering domain is a constant which belongs to the derived set of \(P(f)\) (see [BHKMT]), which is empty by (iii) in our case. Now it is easy to complete the proof by using these observations and Theorem B. \(\square\)

**Corollary D** Let \(f\) be a semi-hyperbolic (transcendental) entire function with \(J(f) \neq \mathbb{C}\).

1. \(\text{Leb}(J(f)) = 0\) if and only if \(\text{Leb}(J(f) \cap I(f)) = 0\), where \(I(f) := \{z \mid f^n(z) \to \infty\}\).            
2. If \(\text{Leb}(J(f)) > 0\), then \(f^n(z) \to \infty \ (n \to \infty)\) for a.e. \(z \in J(f)\).

(Proof): Since \(f\) is semi-hyperbolic, we have \(Z = \emptyset\) by Theorem A. Also (ATZ) holds from Theorem B, because we assume that \(J(f) \neq \mathbb{C}\). This means that \(f^n(z) \to \infty \) for a.e. \(z \in J(f)\). Now it is obvious to see that (1) and (2) hold. \(\square\)

We can also show the following theorems on the dynamics of semi-hyperbolic transcendental entire functions (see, [K3]).

**Theorem E** Let \(f\) be a transcendental entire function with \(#\text{sing}(f^{-1}) < \infty\) and assume that

(a) \(f\) is semi-hyperbolic,   
(b) \(P(f) \cap J(f)\) is bounded,   
(c) \(J(f) \neq \mathbb{C}\).

Then

1. \(f^n(z) \to \infty\) and \(d_E(f^n(z), P(f)) \to \infty\) for almost every \(z \in J(f)\), where \(d_E\) denotes the Euclidean distance in \(\mathbb{C}\).
2. Moreover, if \(J(f)\) is thin at \(\infty\), then \(\text{Leb}(J(f)) = 0\). \(\square\)

**Theorem F** Let \(f\) be a transcendental entire function and assume that

(a) \(f\) is uniformly semi-hyperbolic (i.e. we can take \(N \in \mathbb{N}\) in the definition of semi-hyperbolicity independently of \(z \in J(f)\)),
(b) \(P(f) \cap J(f)\) is bounded,   
(c) \(d_E(J(f), P(f) \cap F(f)) > 0\),   
(d) \(J(f) \neq \mathbb{C}\).

Then the same conclusion as in Theorem E holds. \(\square\)

### 4. Examples

Let us conclude this paper by showing some examples including various examples of metric global attractors. First three examples can be found in [B].

**Example** (1) \(f(z) = e^z\):

In this case, \(\text{sing}(f^{-1}) = \{0\}\) and 0 is an asymptotic value which satisfies \(f^n(0) \to \infty\), so we have \(J(f) = \mathbb{C}\). However, (ATZ) holds, that is, \(\omega(z) \subseteq \mathbb{C} = \{f^n(0)\}_{n=0}^\infty \cup \{\infty\}\) for a.e. \(z \in J(f) = \mathbb{C}\). This result is due to Lyubich ([L]) and independently by Rees ([R]).
(2) $f(z) = 2\pi i e^z$:
In this case, $\text{sing}(f^{-1}) = \{0\}$ and 0 is an asymptotic value which satisfies $f(0) = 2\pi i$, $f(2\pi i) = 2\pi i$, so 0 is preperiodic. Then it follows that $J(f) = \mathbb{C}$. Hence we can apply Corollary C. Since it is known that $\text{Leb}(I(f)) = 0$ ([Mc]), where $I(f) = \{z \mid f^n(z) \to \infty\}$, it follows that (FOD) (and also (IR) and (ERG)) holds.

(3) $f(z) = \sin \pi z$:
In this case, $\text{sing}(f^{-1}) = \{1, -1\}$ and both singular values are critical values which are preperiodic. So we can apply Corollary C again. Since it is known that $\text{Leb}(I(f)) > 0$ ([Mc]), (MGA) holds, that is, $\{\infty\}$ is a metric global attractor. Nevertheless $J(f) = \mathbb{C}$ holds, that is, $f$ is chaotic on the hole plane $\mathbb{C}$.

(4) $f(z) = z - 1 + e^z$:
This is the famous Fatou's example (1923), which has a completely invariant Baker domain. Since $\text{sing}(f^{-1}) = \{f((2n + 1)\pi i) \mid n \in \mathbb{Z}\} = \{(2n + 1)\pi i - 2 \mid n \in \mathbb{Z}\}$ which consists only of critical values and all these values tend to $\infty$ under the iterate of $f$, we can apply Corollary C. Since $J(f) \neq \mathbb{C}$ as we already know, it follows that $\{\infty\}$ is a metric global attractor.

(5) In [KS], we constructed an $f$ with doubly-connected wandering domains. This $f$ satisfies that $f^2(c) = 0$ for every critical point $c$ and 0 is a repelling fixed point. Also $f$ has no asymptotic value, since $f$ has a doubly-connected wandering domain. Hence from Theorem B, we conclude that $\{\infty\}$ is a metric global attractor. We do not know whether $\text{Leb}(J(f)) > 0$ or not. Also we do not know whether there exists a simply connected wandering domain for this $f$. But the above result implies that even if $\text{Leb}(J(f)) > 0$, it holds that $f^n(z) \to \infty$ for a.e. $z \in J(f)$, which means that the dynamics on $J(f)$ is trivial at least from measure theoretical point of view. Also even if there exists a simply connected wandering domain $V$, it holds that $f^n|_V \to \infty$.

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