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TOROIDAL GROUPS WITHOUT NONCONSTANT MEROMORPHIC FUNCTIONS

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Abstract. We characterize toroidal groups without nonconstant meromorphic functions. This characterization was obtained by Abe and Kopfermann [1]. We get this by studying the space of holomorphic sections on toroidal groups and give a method of constructing such toroidal groups. Proofs will appear elsewhere.

1. TOROIDAL GROUPS

Definition 1. A connected complex Lie group $X$ is called a toroidal group if $H^0(X,\mathcal{O}) = \mathbb{C}$.

By the definition, $X$ is a complex abelian Lie group and there exists a discrete subgroup $\Gamma$ of $\mathbb{C}^n$ such that $X \cong \mathbb{C}^n/\Gamma$, where $n$ is a complex dimension of $X$. Put $r = \text{rank} \, \Gamma$, then $n + 1 \leq r \leq 2n$ and there exist $\mathbb{R}$-linearly independent vectors $\lambda_1, \ldots, \lambda_r \in \mathbb{C}^n$ ($n + 1 \leq r \leq 2n$) satisfying $\Gamma = \mathbb{Z}\{\lambda_1, \ldots, \lambda_r\}$. A matrix $P = \{\lambda_1, \ldots, \lambda_r\}$ is called a period matrix for $\mathbb{C}^n/\Gamma$. Put $\mathbb{R}_\Gamma = \mathbb{R}\{\lambda_1, \ldots, \lambda_r\}$, then $K := \mathbb{R}_\Gamma/\Gamma$ is a maximal compact subgroup of $\mathbb{C}^n/\Gamma$. Let $C_\Gamma := \mathbb{R}_\Gamma \cap \sqrt{-1}\mathbb{R}_\Gamma$ be the maximal complex subspace of $\mathbb{R}_\Gamma$. By the result of Matsushima and Morimoto [2], a complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a Stein group if and only if $C_\Gamma = \{0\}$. Hence for a toroidal group $\mathbb{C}^n/\Gamma$, we have $\dim C_\Gamma > 0$.

Definition 2. A toroidal group $\mathbb{C}^n/\Gamma$ is said to be type $q$, if $\dim C_\Gamma = q$ ($q > 0$).

Then, a toroidal group $\mathbb{C}^n/\Gamma$ is type $q$ if and only if $\text{rank} \, \Gamma = n + q$.

Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type $q$. By a suitable linear change of $\mathbb{C}^n$, a period matrix $P$ can be written as $P = [I_n, V]$, where $I_n = [e_1, \ldots, e_n]$ is an identity matrix and $V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \ldots, v_q]$ is an $n \times q$ matrix.

Proposition 1. A complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if the following condition is satisfied:

$$\text{for any } \sigma \neq 0 \in \mathbb{C}^n, \ i\sigma P \notin \mathbb{Z}^{n+q}. \tag{1}$$

Put $V_1 = [v_{ij}; 1 \leq i, j \leq q]$, and $V_2 = [v_{ij}; q + 1 \leq i \leq n, 1 \leq j \leq q]$. We can assume $\det(\text{Im} \, V_1) \neq 0$. Further we put $v_i = \sqrt{-1}e_i \, (q + 1 \leq i \leq n)$, $V = [v_{ij}; 1 \leq i, j \leq n] = [v_1, \ldots, v_n]$, $A = \text{Re} \, V = [\alpha_{ij}; 1 \leq i, j \leq n]$, $B = \text{Im} \, V = [\beta_{ij}; 1 \leq i, j \leq n] = [\beta_1, \ldots, \beta_n]$, $A_1 = [\alpha_{ij}; 1 \leq i, j \leq q]$, $A_2 = [\alpha_{ij}; q + 1 \leq i \leq n, 1 \leq j \leq q]$, $B_1 = [\beta_{ij}; 1 \leq i, j \leq q]$, and $B_2 = [\beta_{ij}; q + 1 \leq i \leq n, 1 \leq j \leq q]$.

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By the definition, $\beta_1, \ldots, \beta_n$ are $C$-linearly independent. We define two coordinates of $\mathbb{C}^n/\Gamma$ as follows:

$$z = z_1\beta_1 + \cdots + z_n\beta_n$$
$$= t_1e_1 + \cdots + t_ne_n + t_{n+1}v_1 + \cdots + t_{2n}v_n.$$  \hfill (2)

Put $t^1 = t(t_1, \ldots, t_n)$, $t^2 = t(t_{n+1}, \ldots, t_{2n})$, $t^{1'} = t(t_1, \ldots, t_q)$, $t^{1''} = t(t_{q+1}, \ldots, t_n)$, $t^{2''} = t(t_{n+1}, \ldots, t_{n+q})$, $t^{2'''} = t(t_{n+q+1}, \ldots, t_{2n})$.

$$P_\beta^\prime = B^{-1}P = \begin{bmatrix} B_1^{-1} & O & B_1^{-1}V_1 \\ -B_2B_1^{-1} & I_{n-q} & -B_2B_1^{-1}A_1 + A_2 \end{bmatrix},$$
and
$$P_\beta = \begin{bmatrix} O & T_\beta \\ I_{n-q} & R \end{bmatrix},$$
where $T_\beta = [B_1^{-1}, B_1^{-1}V_1]$, and $R = [R_1, R_2] = [-B_2B_1^{-1}, -B_2B_1^{-1}A_1 + A_2]$.

We define the map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by the following commutative diagram

$$\begin{array}{ccc}
z & \longrightarrow & t \\
\downarrow & & \downarrow \\
\sqrt{-1}z & \longrightarrow & Jt.
\end{array}$$

Then $Jt = \begin{bmatrix} B & -A \\ O & I_n \end{bmatrix} \begin{bmatrix} O & -I_n \\ I_n & O \end{bmatrix} \begin{bmatrix} B & -A \\ O & I_n \end{bmatrix}^{-1} \begin{bmatrix} t^1 \\ t^2 \end{bmatrix}$, and $t \in \mathbb{C}_\Gamma$ if and only if $Jt \in \mathbb{R}_\Gamma$.

Then we have

**Proposition 2.** Let $t \in \mathbb{R}_\Gamma$. Then

$$t \in \mathbb{C}_\Gamma \text{ if and only if } t^{1''} + R_1t^{1'} + R_2t^{2'} = 0.$$  

We write $R = [r_1, \ldots, r_{2q}]$, where $r_1, \ldots, r_{2q} \in \mathbb{R}\{e_{q+1}, \ldots, e_n\}$.

Put $\Gamma_\beta = \mathbb{Z}\{e_{q+1}, \ldots, e_n, B^{-1}e_1, \ldots, B^{-1}e_q, B^{-1}v_1, \ldots, B^{-1}v_q\}$, and $\Gamma'' = \mathbb{Z}\{e_{q+1}, \ldots, e_n, r_1, \ldots, r_{2q}\}$, then we have the following

**Proposition 3.** $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if

$$\text{for any } m'' \in \mathbb{Z}^{n-q}\{0\}, \quad t^{m''}R \notin \mathbb{Z}^{2q}.$$  \hfill (3)

Hence we have

**Corollary 1.** $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if

$$\Gamma'' \text{ is dense in } \mathbb{R}^{n-q} = \mathbb{R}\{e_{q+1}, \ldots, e_n\}$$

We define the mapping $\pi_\Gamma : \mathbb{R}_\Gamma \to \mathbb{R}_\Gamma/\Gamma$ by

$$\pi_\Gamma(t) = t(\exp(2\pi\sqrt{-1}t_1), \ldots, \exp(2\pi\sqrt{-1}t_{n+q})).$$

Then

**Corollary 2.** $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if $\pi_\Gamma(\mathbb{C}_\Gamma)$ is dense in $\mathbb{R}_\Gamma/\Gamma$.  

2. Holomorphic sections of complex line bundles on toroidal groups

Let \( X = \mathbb{C}^n/\Gamma \) be a toroidal group of type \( q \), \( L \to X \) a complex line bundle on \( X \) and \( c_1(L) = E \in H^2(X, \mathbb{Z}) \) the first Chern class of \( L \). Then there exists a hermitian form \( H \) on \( \mathbb{C}^n \) satisfying \( \text{Im} \, H|\Gamma \times \Gamma = E \).

Let \( e_\lambda(z) \in H^1(X, \mathcal{O}^*) \) \((\lambda \in \Gamma)\) be the cocycle defining the complex line bundle \( L \). Then we can write
\[
e_\lambda(z) = \alpha(\lambda) \exp \left( \pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda) \right) \exp \left( 2\pi \sqrt{-1} a_\lambda(z) \right),
\]
where \( \alpha : \Gamma \to \mathbb{C}^*_1 = \{z||z| = 1\} \) satisfies \( \alpha(\lambda + \mu) = \alpha(\lambda) \alpha(\mu) \exp \left( \pi \sqrt{-1} E(\lambda, \mu) \right) \), and \( a_\lambda \in H(X, \mathcal{O}) \), namely \( a_\lambda(z) \in H^0(\mathbb{C}^n, \mathcal{O}) \) satisfying
\[
a_\lambda(z + \mu) + a_\mu(z) = a_{\lambda + \mu}(z), \quad \text{and} \quad a_\lambda(z) = a_\lambda(0, \ldots, 0, z_{q+1}, \ldots, z_n).
\]
We note that the above description of holomorphic cocycles are obtained by C. Vogt [7] (cf. [3]).

**Proposition 4.** Let \( X = \mathbb{C}^n/\Gamma \) be a toroidal group of type \( q \) and \( L \) a topologically trivial line bundle over \( X \).

If there exists \( s \neq 0 \in H^0(X, \mathcal{O}(L)) \), then \( L \) is analytically trivial.

Further we have the following

**Proposition 5.** Let \( X = \mathbb{C}^n/\Gamma \) be a toroidal group of type \( q \), \( L \) a complex line bundle over \( X \) and \( c_1(L) = E \neq 0 \).

Suppose there exists a hermitian form \( H \) on \( \mathbb{C}^n \) satisfying

1. \( \text{Im} \, H|\Gamma \times \Gamma = E \), and
2. there exists \( w \in \mathbb{C}_\Gamma \) such that \( H(w, w) < 0 \).

Then we have \( H^0(X, \mathcal{O}(L)) = 0 \).

**Definition 3.** A toroidal group \( X = \mathbb{C}^n/\Gamma \) is said to be a toroidal group with semi-positive hermitian form if \( X \) satisfies the following conditions:

There exist a complex line bundle \( L \) over \( X \) with \( c_1(L) = E \) and a hermitian form \( H \) on \( \mathbb{C}^n \) such that

1. \( \text{Im} \, H|\Gamma \times \Gamma = E \), and
2. \( H|\mathbb{C}_\Gamma \times \mathbb{C}_\Gamma \geq 0 \),
3. there exists \( w \in \mathbb{C}_\Gamma \) such that \( H(w, w) > 0 \).

Let \( X = \mathbb{C}^n/\Gamma \) be a toroidal group with semi-positive hermitian form. Put \( N = \{z \in \mathbb{C}^n|H(z, w) = 0, \forall w \in \mathbb{C}^n\} \) and \( W = \{z \in \mathbb{C}^n|\exists w \in \mathbb{C}^n \text{ s.t. } H(z, w) \neq 0\} \).

Then \( \mathbb{C}^n = N \oplus W \). For \( z \in \mathbb{C}^n \), write \( z = z_N + z_0 \), where \( z_N \in N \) and \( z_0 \in W \).

Let \( e_\lambda(z) \) be the cocycle which defines \( L \) satisfying (4) and (5). For any \( \lambda_N \in N \cap \Gamma \), we have \( e_{\lambda_N}(z) = \alpha(\lambda_N) \exp \left( 2\pi \sqrt{-1} a_{\lambda_N}(z) \right) \).

Let \( L_N \) be the line bundle on \( \mathbb{C}^n/\Gamma \cap N \) defined by the cocycle \( \{e_{\lambda_N}(z)\} \). Clearly, \( L_N \) is topologically trivial.
Lemma 1. Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type $q$ with semi positive hermitian form $H$ and a complex line bundle $L$ over $X$ and $c_1(L) = E$.

Then there exists a semi-positive hermitian form $H'$ with $N \subset C\Gamma$.

Then we get the following

Theorem 1. Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group with a complex line bundle $L$ on $X$.
Then there exists a non constant $s(z) \in H^0(X, \mathcal{O}(L))$ if and only if $X$ is a toroidal group with semipositive hermitian form such that $L_N \rightarrow \mathbb{C}^n/N \cap \Gamma$ is analytically trivial.

3. Toroidal groups without nonconstant meromorphic functions

Let $f(z)$ be a meromorphic function on $X$, then $f(z)$ has a period group $G(f)$ which satisfies $\Gamma \subset G(f)$.

Then there exist $h, \ k \in H^0(\mathbb{C}^n, \mathcal{O})$ such that $(h, k) = 1$ and $f(z) = \frac{h(z)}{k(z)}$, for all $z \in \delta(f)$, where $\delta(f) =$the set of regular points of $f(z)$. For each $\lambda \in \Gamma$, set for all $z \in \mathbb{C}^n$, $h_\lambda(z) := h(z + \lambda)$, and $k_\lambda(z) := k(z + \lambda)$. For all $z \in \delta(f)$, \[ \frac{h(z)}{k(z)} = f(z) = f(z + \lambda) = \frac{h_\lambda(z)}{k_\lambda(z)} \]

Since $(h_\lambda, k_\lambda) = 1$, there exist $v_\lambda \in H^0(\mathbb{C}^n, \mathcal{O})$ such that $h(z + \lambda) = h(z) \exp(v_\lambda(z))$, and $k(z + \lambda) = k(z) \exp(v_\lambda(z))$.

Hence $\exp(v_{\lambda_1 + \lambda_2}(z)) = \exp(v_{\lambda_2}(z + \lambda_1)) \exp(v_{\lambda_1}(z))$.

Namely, $\{\exp(v_\lambda(z))\}$ defines a line bundle $L$ on $X$ and $h(z), \ k(z) \in H^0(X, \mathcal{O}(L))$.

Consider the exact sequence\[ \rightarrow H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathcal{O}) \rightarrow H^2(X, \mathcal{O}) \rightarrow \]

Definition 4. $\text{NS}(X) := c_1H^1(X, \mathcal{O}^*)$ is called a Néron-Severi group.

Suppose $E \in \text{NS}(X)$, then $\iota(E) = 0$ in $H^2(X, \mathcal{O})$. Since $E$ is a $\mathbb{Z}$-valued skew-symmetric $(n + q, n + q)$-matrix, we write \[ E = [E_{ij}; 1 \leq i, j \leq n + q] = \begin{bmatrix} E_1 & E_2 \\ -^tE_2 & E_3 \end{bmatrix}, \]

where $E_1 \in \mathbb{Z}^{n \times n}$, and $E_3 \in \mathbb{Z}^{q \times q}$.

The following theorem characterizes the Néron-Severi group on a toroidal group ([6]).

Theorem 2. Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type $q$ with a period matrix $P = [I_n, V]$. Then $E \in H^2(X, \mathcal{O})$ belongs to $\text{NS}(X)$ if and only if \[ ^tVE_1V + ^tE_2V - ^tVE_2 + E_3 = 0 \]

Then we get the following
Theorem 3. Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group. If the Néron-Severi group $\text{NS}(X) = 0$, there are no nonconstant meromorphic function on $X$.

Definition 5. Let $\alpha$ and $\beta$ are two algebraic numbers. $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are said to be linearly disjoint over $\mathbb{Q}$ if a ring homomorphism

$$\mathbb{Q}(\alpha) \otimes \mathbb{Q}(\beta) \ni x \otimes y \mapsto xy \in \mathbb{Q}(\alpha, \beta)$$

is an isomorphism.

Theorem 4. Let $X = \mathbb{C}^n/\Gamma$ be a complex abelian Lie group with the period matrix $P = [I_n, V]$ which satisfies the following conditions.

1. $V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q]$

2. $q \geq 2$ and $\beta_{ij}$ are algebraic numbers with $[\mathbb{Q}(\beta_{ij}) : \mathbb{Q}] \geq 2$.

3. $\mathbb{Q}(\beta_{ij})$ and $\mathbb{Q}(\beta_{kl})$ are linearly disjoint for each $(i, j) \neq (k, l)$.

Then $\text{NS}(X) = 0$.

Hence $X$ is a toroidal group which has no non-constant meromorphic functions on it.

The followings are examples of toroidal groups without non-constant meromorphic functions, which satisfy the conditions (1), (2) and (3) of Theorem 4. We note that Theorems 4 is a generalization of example of Siegel (Example 1) to toroidal groups.

Example 1. (Siegel [4]) $X = \mathbb{C}^2/\Gamma$,

$$P = \begin{bmatrix}
1 & 0 & \sqrt{-2} & \sqrt{-5} \\
0 & 1 & \sqrt{-3} & \sqrt{-7}
\end{bmatrix}$$

Example 2. $X = \mathbb{C}^3/\Gamma$,

$$P = \begin{bmatrix}
1 & 0 & 0 & \sqrt{-2} & \sqrt{-7} \\
0 & 1 & 0 & \sqrt{-3} & \sqrt{-11} \\
0 & 0 & 1 & \sqrt{-5} & \sqrt{-13}
\end{bmatrix}$$

References


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