BIRATIONAL EMBEDDING OF ALGEBRAIC PLANE CURVES BY MIXED PLURICANONICAL MAPS

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ABSTRACT. In this paper, we study pairs \((S,D)\) where \(S\) is a projective nonsingular rational surface and \(D\) a nonsingular curve on \(S\). Suppose that the Kodaira dimension of \((S,D)\) is 2, which is defined to be \(\kappa(D+K_S, S)\), \(K_S\) being a canonical divisor on \(S\). By \(Z_m\) denoting a divisor \(m(2K_S+D)\), if \(\sigma \geq 5\) then \(Z_m\) is nef and big, where \(\sigma\) is the covering degree of \(C\) onto the base curve of \(S\).

Our purpose is to show the following result: Under the hypothesis that the Kodaira dimension of \((S,D)\) is 2, \(\sigma \geq 5\) and \((S,D)\) is minimal,

1. if \(m \geq 2\) then \(|Z_m|\) has no base points.
2. If \(m \geq 3\) then the rational map \(\Psi_m\) associated with \(|Z_m|\) is a birational morphism onto its image.

Proofs will appear elsewhere.

1. INTRODUCTION

First, we start with recalling well known facts concerning algebraic curves defined over \(\mathbb{C}\). Let \(C\) be an algebraic curve of general type (i.e., \(g = \text{genus} > 1\)). Associated with the canonical divisor \(K_C\), one has a rational map \(\Phi_K\), called the canonical map of \(C\).

1. If \(C\) is not hyperelliptic, then \(\Phi_K\) is one-to-one onto its image.
2. If \(C\) is hyperelliptic, then \(\Phi_K : C \to \mathbb{P}^1\) is a double cover. But in this case, \(\Phi_{2K}\) is one-to-one onto its image.

For algebraic surfaces, the next result (see [8], [1]) gives the best estimate for birationality of pluricanonical maps. Let \(S\) be an algebraic minimal surface of general type. Then the canonical divisor \(K_S\) is nef (numerically effective) and big, in other words, \(K_S^2 > 0\).

Define \(P_m(S)\) to be \(\dim |mK_S| + 1\), that is called \(m\)-genus. Then by Kodaira [7], \(H^1(S, \mathcal{O}(mK_S)) = 0\) if \(m \geq 2\). Hence,

\[
P_m(S) = \frac{m(m-1)}{2}K_S^2 + 1 - q + p_g.
\]

Here, \(p_g\) denotes the geometric genus and \(q\) means the irregularity of \(S\). Associated with \(mK_S\) we have rational maps denoted by \(\Phi_m\), called pluricanonical maps.

**Theorem 1.1** (Kodaira. Bombieri). (1) If \(m \geq 4\), then \(|mK_S|\) has no base points.
(2) If \(m \geq 5\), then \(\Phi_m\) is a birational map onto its image.

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The purpose here is to transplant these results into Cremonian geometry. In Cremonian geometry, pairs \((S, C)\) of nonsingular rational surfaces \(S\) and curves \(C \subset S\) are objects of the study. For pairs \((S, C)\) where \(C\) is nonsingular, \(D\) stands for \(C\).

In the theory of pairs \((S, D)\), \(Z = K_S + D\) and \(2Z - D = 2K_S + D\) are fundamental tools. Moreover, \(mK_S + aD(1 < m/a < 2)\) are sometimes quite useful.

Given pairs \((S, C)\) and \((S_1, C_1)\), if there exists a birational map \(h : S \to S_1\) such that the proper transform \(h[C]\) coincides with \(C_1\), i.e., \(h[C] = C_1\) then they are said to be birationally equivalent.

\[ P_m[D] = \dim \{mZ\} + 1 \] are called logarithmic plurigenera, which are birational invariants in the above sense.

Note that if \(S = \mathbb{P}^2\), then birational maps between \((S, C)\) and \((S, C_1)\) are Cremona transformations, by which \(C\) is transformed into \(C_1\). In this case, \(C\) is an algebraic plane curve. Thus, pairs \((S, C)\) are regarded as generalization of algebraic plane curves.

Suppose that \(m \geq a \geq 1\). Then \(P_{m,a}[D] = \dim \{mK_S + aD\} + 1\) are called mixed plurigenera, which depend on \(S\) and \(D\). Associated with \(mK_S + aD\), one has the rational map, called mixed pluricanonical map. Note that \(P_{m,m}[D]\) turns out to be \(P_m[D]\) and \(P_m[D] \sim cm^c(c > 0)\) as \(m \to \infty\), by which Kodaira dimension of the pair \((S, D)\) is defined to be \(\kappa\), denoted \(\kappa[D]\).

Structures of pairs \((S, D)\) with \(\kappa[D] < 2\) have been completely determined (see [5]). Hence, we shall study pairs \((S, D)\) with \(\kappa[D] = 2\) and prove a counterpart of Kodaira-Bombieri’s theorem.

2. Some basic results

2.1. Minimal models. A non-singular pair \((S, D)\) is said to be relatively minimal, whenever \(D \cdot E \geq 2\) for any \((-1)\) curve \(E\) on \(S\) such that \(E \neq D\). It is shown that if \((S, D)\) is relatively minimal and \(\kappa[D] = 2\) then \((S, D)\) is actually minimal, in other words, any birational map from \((S_1, D_1)\) to \((S, D)\) turns out to be regular (see [4]). In particular, any birational maps from \((S, D)\) to itself become automorphisms.

**Proposition 2.1.** Suppose that \((S, D)\) is minimal.

1. If \(g = g(D) > 0\) then \(Z = K_S + D\) is nef. Moreover, when \(\kappa[D] = 2\), \(Z\) is big.
2. If \(g = 0\) and \(\kappa[D] = 2\) then \(D^2 \leq -5\) and letting \(\beta\) denote \(-D^2\), \(Z\beta = Z - 2\beta D\) is nef and big.

Minimal pairs are obtained from some kind of singular models, namely, \# minimal pair which will be defined below. Any \(\mathbb{P}^1\)- bundle over \(\mathbb{P}^1\) has a section \(\Delta_{\infty}\) with negative self intersection number, which is denoted by a symbol \(\Sigma_B\), where \(-B = \Delta_{\infty}^2\) if \(B > 0\). \(\Sigma_B\) is said to be a Hirzebruch surface of degree \(B\) after Kodaira.

The Picard group of \(\Sigma_B\) is generated by a section \(\Delta_{\infty}\) and a fiber \(F_c = \rho^{-1}(c)\) of the \(\mathbb{P}^1\)- bundle, where \(c \in \mathbb{P}^1\) and \(\rho : \Sigma_B \to \mathbb{P}^1\) is the projection.

Let \(C\) be an irreducible curve on \(\Sigma_B\). Then \(C \sim \sigma \Delta_{\infty} + \epsilon F_c\), for some \(\sigma\) and \(\epsilon\). Here the symbol \(\sim\) means the linear equivalence. We have \(C \cdot F_c = \sigma\) and \(C \cdot \Delta_{\infty} = \epsilon - B \cdot \sigma\).

Hereafter, suppose that \(C \neq \Delta_{\infty}\). Thus \(C \cdot \Delta_{\infty} \geq 0\) and hence, \(\epsilon \geq B\sigma\).

If \(B > 0\) then \(\Delta_{\infty}^2 = -B < 0\) and such a section \(\Delta_{\infty}\) is uniquely determined.
For a surface $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, we get $F_c \sim \mathbb{P}^1 \times \text{point}$ and $\Delta_\infty \sim \text{point} \times \mathbb{P}^1$, and in this case $e \geq \sigma$ is assumed. Thus $\sigma$ and $e$ are uniquely determined for a given curve $C$ on $\Sigma_B$.

2.2. Types of pairs and # minimal pairs. By $\nu_1, \nu_2, \ldots, \nu_r$ we denote the multiplicities of all singular points (including infinitely near singular points) of $C$ where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

The symbol $[\sigma * e, B; \nu_1, \nu_2, \ldots, \nu_r]$ is said to be the type of $(\Sigma_B, C)$.

Assume that $\sigma \geq 2\nu_1$ and $e \geq \sigma + B\nu_1$. Moreover, if $B = \nu_1 = 1$ then assume $e - \sigma > 1$. When the above conditions are satisfied, the pair $(\Sigma_B, C)$ is said to be # minimal.

Theorem 2.1. Any minimal pair $(S, D)$ is obtained from a # minimal pair $(\Sigma_B, C)$ by resolving singularities of $C$ by blowing ups, if it is not isomorphic to $(\mathbb{P}^2, C_d)$, $C_d$ being a nonsingular curve.

Therefore, let the type of the pair $(S, D)$ mean that of # minimal $(\Sigma_B, C)$.

Note that if $\sigma \geq 4$, then $(2Z - D) \cdot Z/2 \geq 1$. Hence, defining an invariant $A$ to be $(2Z - D) \cdot Z/2$, we have $A = Z^2 - g + 1$ and $P_{2,1}[D] = A + 1$. Pairs with small mixed plurigenera are easily enumerated. For example, we have the next result (see [6]):

Theorem 2.2. Suppose that $P_2[D] = 2g \geq 2$. (In other words, $Z^2 = 1$)

1. If $g = 3$, then $D^2 = 16$ and the type of the curve is $[4; 1]^1$.
2. If $g = 2$, then $D^2 = 4$ and the type is $[4 * 4, 0; 2]$ or its associates.
3. If $g = 1$, then
   a. If $D^2 = -2$, then the type is $[8 * 8, 0; 4^2, 3^2]$ or its associates.
   b. If $D^2 = -3$, then the type is $[6 * 6, 0; 3^2, 2^3]$ or its associates.
   c. If $D^2 = -4$, then the type is $[4 * 5, 0; 2^{11}]$ or its associates.

3. MAIN RESULTS

3.1. Statement of the main theorem. Note that if $\sigma \leq 4$, then $\nu_1 = 1, 2$ and the structures of $(S, D)$ are easily investigated ([6]). So, we assume $\kappa[D] = 2$ and $\sigma \geq 5$. Then $2Z - D$ is nef and big.

Defining $Z_m$ to be $m(2Z - D)$, one has the rational map associated with $|Z_m|$, denoted by $\Psi_m$.

Theorem 3.1. (1) If $m \geq 2$ then $|m(2Z - D)|$ has no base points.
   (2) If $m \geq 3$ then $\Psi_m$ is a birational morphism onto its image.

This will be proved by applying various kinds of vanishing theorems of cohomology, namely, Kawamata – Viehweg vanishing theorem and Ramanujam’s strong vanishing theorem (see [1]). We shall imitate the argument of Bombieri [1]. Actually, the following lemmas and propositions are indispensable.

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1. This implies that the curve is a nonsingular plane curve of degree 4.
2. Types $[4 * 6, 1; 2^7]$ and $[4 * 8, 2; 2^7]$ are associates for the type $[4 * 4, 0; 2^7]$
3.2. Formula for mixed plurigenera.

Lemma 3.1. For positive numbers $m$ and $a$ such that $1 < m/a < 2$, let $W_\varepsilon$ be a \(\mathbb{Q}\)-divisor $mK_S + aD - \varepsilon D$, where $\varepsilon$ is a positive rational number $< 1$. Then $W_\varepsilon$ is nef and big, whenever $g > 0$ and $2\varepsilon < 2a - m$. Moreover, if $g = 0$, $2\varepsilon < 2a - m$ and $-2m - (m - a + \varepsilon)D^2 > 0$ then $W_\varepsilon$ is nef and big.

Proposition 3.1. For $m \geq 2$,

\[
P_{2m,m}[D] = \frac{1}{2}m(m-1)Q + mA + 1,
\]

\[
P_{2m-1,m}[D] = \frac{(m-1)(m-2)}{2}Q + (3m-4)A + Z^2 + 1
\]

where $Q = (2K_S + D)^2$, $A = Z^2 - g + 1$.

In particular, if $m = 3$ then $P_{3,3}[D] = Q + 5A + Z^2 + 1 \geq 7$. Moreover, $P_{3,2}[D] = 2A + Z^2 + 1 \geq 4$.

For example, the pair $(S, D)$ with type $[10 \times 11, 0; 5^9]$ satisfies that $Q = 3, Z^2 = 0, A = 1, P_{3,2}[D] = 3$ and $P_{3,3}[D] = 9$.

Lemma 3.2. $L_m = (m-1)(2Z - D) + Z$ is nef and big if $m \geq 2$ except in the case when $g = 0$ and $D^2 = -5$. In the exceptional case, we assume $m \geq 3$. Then $L_m$ is nef and

\[
L_m^2 = ((m-1)(2Z - D) + Z)^2
= (m-1)^2Q + 4(m-1)A + Z^2
\geq (m-1)^2 + 4(m-1) + 1
= (m-1)(m+3) + 1.
\]

In particular, if $m \geq 2$, then $L_m^2 \geq 6$. Moreover, $L_m^2 = 6$ implies that $Q = 1, A = 1, Z^2 = 1$ and the type is $[6 \times 6, 0; 3^7, 2^3]$, where $g = 0$, $F_{3,2}[D] = 4$. Further, if $m \geq 3$, then $L_m^2 \geq 13$.

3.3. Connectedness of divisors. In general, given $i > 0$, an effective divisor $L$ is said to be $i-$ connected, if $X_1 \cdot X_2 \geq i$ for any nonzero effective divisors $X_1$ and $X_2$ such that $X_1 + X_2 \sim L$.

In that follows, given $m \geq 2$, let $L_m$ denote $m(2Z - D) - K_S$, which is $(m-1)(2Z - D) + Z$. In the case when $g = 0$ and $D^2 = -5$, assume $m \geq 3$. Then $L_m$ is nef and big. By Proposition 3.1, $\dim |L_m| + 1 = P_{2m-1,m}[D] \geq 4$. Hence, we may suppose that $L_m$ is an effective divisor.

Proposition 3.2. If $m \geq 2$ then $L_m$ is 2-connected and $L_m^2 \geq 6$.

For a point $x_0$ on $S$, by blowing up at $x_0$, we have a birational morphism $\mu : S_1 \to S$. Letting $E_0 = \mu^{-1}(x_0)$ we have the following exact sequence:

\[
0 \to \mathcal{O}(\mu^*(Z_m) - E_0) \to \mathcal{O}(\mu^*(Z_m)) \to \mathcal{O}_{E_0} \to 0
\]

Then

\[
0 \to H^0(S_1, \mathcal{O}(\mu^*(Z_m) - E_0)) \to H^0(S_1, \mathcal{O}(\mu^*(Z_m))) \to H^0(E_0, \mathcal{O}_{E_0})
\to H^1(S_1, \mathcal{O}(\mu^*(Z_m) - E_0))
\]
Since $H^0(\mathcal{O}_{E_0}) = \mathbb{C}$. if $H^1(S_1, \mathcal{O}(\mu^*Z_m - E_0)) = 0$, then $x_0$ is not a base point of $|Z_m|$. Thus, we shall show the vanishing of $H^1(S_1, \mathcal{O}(\mu^*Z_m - E_0))$.

First, by Serre duality, $H^1(S_1, \mathcal{O}(\mu^*Z_m - E_0)) = H^1(S_1, \mathcal{O}(K_{S_1} - \mu^*Z_m + E_0))$. However,

$$K_{S_1} - \mu^*Z_m + E_0 = 2E_0 - \mu^*(Z_m - K_S) = 2E_0 - \mu^*(L_m).$$

In order to show that $H^1(S_1, \mathcal{O}(2E_0 - \mu^*(L_m))) = 0$, by Ramanujam’s vanishing theorem, it suffices to prove that $\mu^*(L_m) - 2E_0$ is linearly equivalent to an effective divisor, which is 1-connected with positive self-intersection number.

Since dim $|L_m| \geq 3$, there exists an effective divisor $X \in |L_m|$ which has the multiplicity 2 at $x_0$. Then we get an effective divisor $X'$ on $S_1$ such that $\mu^*(X) \sim X' + 2E_0$. Therefore, $\mu^*(L_m) - 2E_0 \sim X'$. Since $L_m$ is 2-connected, $X'$ is 1-connected and $X'^2 = L_m^2 - 4 \geq 2$. Thus,

$$H^1(S_1, \mathcal{O}(2E_0 - \mu^*(L_m))) = H^1(S_1, \mathcal{O}(-X')) = 0.$$

Hence, $|Z_m|$ has no base points if $m \geq 2$.

Furthermore, by similar but more complicated argument, we can complete the proof of Theorem 3.1.

References

[8] K. Kodaira, Pluricanonical systems on algebraic surfaces of general type II.,in Iitaka’s web page.⁴

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³http://www-cc.gakushuin.ac.jp/%7E851051/iitaka1.htm
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