SOME GEOMETRIC MAPPINGS ON THE OPEN UNIT BALL

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ABSTRACT. Let $B$ be the unit ball of a reflexive complex Banach space. We obtain growth and covering theorems for some holomorphic mapping on $B$ with parametric representaion, and consider various examples.

1. INTRODUCTION

Let $(E, \| \cdot \|)$ denote a complex Banach space equipped with the norm $\| \cdot \|$. Let $B_r = \{ z \in \mathbb{C}^n : \| z \| < r \}$ for $r > 0$ and let $B = B_1$. In the case of one complex variable, $B_r$ is denoted by $\Delta_r$ and $\Delta_1$ by $\Delta$. Let $G$ be an open subset in $E$ and let $H(G)$ denote the set of holomorphic mappings from $G \subset E$ into $E$.

By $L(E, F)$, we denote the space of all continuous linear operators from a complex Banach space $E$ into a complex Banach space $F$ with the standard operator norm. Let $\text{Id}$ denote the identity in $L(E, F)$.

If $f \in H(B_r)$, we say that $f$ is normalized if $f(0) = 0$ and $Df(0) = \text{Id}$. Let $N(B_r)$ be the set of all normalized locally biholomorphic mappings in $H(B_r)$. A holomorphic mapping $f : B \to F$ is said to be starlike if $f$ is univalent, $f(0) = 0$ and $tf(B) \subset f(B)$ ($0 \leq t \leq 1$). The set of normalized starlike mappings of $B_r$ is denoted by $S^*(B_r)$.

Suffridge [Su1], [Su2] and Gurganus [Gu] gave some characterizations of the starlikeness for a holomorphic mapping on the unit ball in a complex Banach space. There are many results concerning the estimates of growth for starlike mappings (see [Ba-Fi-Go], [Go1], [Gu] and [Li-Li]).

The purpose of the present study is to give the condition whereby holomorphic mappings satisfy similar estimates of the growth theorem.

2. PARAMETRIC REPRESENTATION

For each $z \in E \setminus \{0\}$, let

$$T(z) = \{ \varphi \in L(E, E) : \varphi(z) = \| z \|, \| \varphi \| = 1 \}.$$

This set $T(z)$ is nonempty by the Hahn-Banach theorem.

We obtain the following theorem from Theorem 1 of Suffridge [Su2]. This theorem gives an analytical characterization of starlike mappings of the open ball in a complex Banach space.

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Theorem 2.1. Let \( f : B \to F \) be a normalized locally biholomorphic mapping. Then \( f \) is starlike if and only if \( \Re \{ \phi (|Df(x)|^{-1} (f(x))) \} > \) 0 for \( x \in B, x \neq 0 \) and \( \phi \in T(x) \).

We recall that a mapping \( f : B \times [0, \infty) \to E \) is called a Loewner chain if the following conditions hold:

(i) \( f(\cdot, t) \) is univalent on \( B, f(0, t) = 0, DF(0, t) = e^t \text{Id} \) for \( t \geq 0 \);
(ii) there exists a univalent Schwarz mapping \( v = v(z, s, t) \) (i.e. \( v \in H(B), v(0) = 0, \) and \( |v(z)| < 1, z \in B \)) such that \( f(z, s) = f(v(z, s, t), t) \) for \( z \in B \) and \( 0 \leq s \leq t < \infty \).

The above Schwarz mapping \( v \) is called the transition mapping associated to \( f(z, t) \). It is known that starlikeness can be characterized in terms of Loewner chains: \( f \in S^*(B) \) if and only if \( f(z, t) = e^t f(z), z \in B, t \geq 0, \) is a Loewner chain [Pf-Su].

Now we set
\[ \mathcal{M} = \{ p \in H(B) : p(0) = 0, Dp(0) = \text{Id}, \Re \phi(p(z)) > 0, z \in B \setminus \{0\}, \phi \in T(z) \}. \]

As in Kohr [Ko], we shall consider properties of the corresponding solutions of the Loewner differential equation.

Definition 2.2. Let \( f : B \to E \) be a normalized locally holomorphic mapping. We say that \( f \) has parametric representation if there exists a mapping \( h = h(z, t) \) which satisfies the following conditions:

(i) \( h(\cdot, t) \in \mathcal{M} \) for \( t > 0 \);
(ii) \( h \) is continuous on \( B \times [0, \infty) \),

such that
\[ f(z) = \lim_{t \to \infty} e^t v(z, t) \]
for \( z \in B \), where \( v = v(z, t) \) is the unique solution of the initial value problem
\[ \frac{\partial v}{\partial t} = -h(v, t), \quad t \geq 0, v(z, 0) = z, \]  \hspace{1cm} (2.1)
for all \( z \in B \).

Let \( S^0(B) \) be the set of all mappings which have parametric representation on \( B \). Then \( S^0(B) \subset N(B) \) (see [Gr-Ha-Ko]). The following result is due to [Ha-Ko].

Theorem 2.3. Let \( f : B \times [0, \infty) \to E \) be a continuous mapping such that \( f(\cdot, t) \in H(B), f(0, t) = 0, DF(0, t) = e^t \text{Id} \) for each \( t \geq 0 \) and \( f(z, \cdot) \) is differentiable on \( [0, \infty) \) for each \( z \in B \), and there exist \( r \in (0, 1), t_0 \geq 0 \) and \( M > 0 \) such that
\[ \|f(z, t)\| \leq e^{t_0} M, \quad \|z\| \leq r, t \geq t_0. \]  \hspace{1cm} (2.2)

We assume that there exists a mapping \( h = h(z, t) \) satisfies the conditions (i) and (ii) of Definition 3.1 such that
\[ \frac{\partial f}{\partial t} = DF(z, t)h(z, t), \quad z \in B, t \geq 0. \]  \hspace{1cm} (2.3)

Then \( f(z, t) \) is a Loewner chain and \( \lim_{t \to \infty} e^t v(z, s, t) = f(z, s) \) for all \( z \in B, s \geq 0, \) where \( v = v(z, s, t) \) is the solution of the initial value problem (2.1). Hence, if \( f(z) = f(z, 0) \) is locally biholomorphic on \( B \), then \( f(z) = f(z, 0) \in S^0(B) \).
According to Pfaltzgraff-Suffridge [Pf-Su], we say that a normalized locally biholomorphic mapping \( f \in H(B) \) is said to be close-to-starlike if there exists a mapping \( g \in S^*(B) \) such that
\[
\Re l_z([Df(z)]^{-1}g(z)) > 0
\]
for all \( z \in B \setminus \{0\} \) and \( l_z \in T(z) \). Let \( C(B) \) denote the set of all close-to-starlike mappings on \( B \). It is known that every mapping \( f \in C(B) \) is univalent on \( B \). Moreover, close-to-starlikeness can also be characterized in terms of Loewner chains, in the sense that \( f \in C(B) \) if and only if there exists a mapping \( g \in S^*(B) \) such that \( f(z,t) = f(z) + (e^t - 1)g(z) \) is a Loewner chain.

3. GROWTH RESULTS FOR MAPPINGS IN \( S^0_{k+1}(B) \)

Let \( B \) be the unit ball in a reflexive complex Banach space, and let \( k \) be a positive integer. Then \( f \in H(B) \) is said to be \( k \)-fold symmetric if the image of \( f \) is unchanged when it multiplied by the scalar complex number \( \exp(2\pi i/k) \). We say that \( z = 0 \) is a zero of order \( k \) of \( f(z) \) if \( f(0) = \ldots , D^{k-1}f(0) = 0 \) and \( D^k f(0) \neq 0 \). We denote by \( S^0_{k+1}(B) \) the subset of \( S^0(B) \) consisting of mappings \( f \) for which there exists a Loewner chain \( f(z,t) \) such that \( \{ e^{-t}f(z,t) \}_{t \geq 0} \) is a normal family on \( B \), \( f = f(\cdot,0) \) and \( z = 0 \) is a zero of order \( k + 1 \) of \( e^{-t}f(z,t) - z \) for each \( t \geq 0 \).

**Lemma 3.1.** Let \( f \in N(B) \). If \( f \) is \( k \)-fold symmetric and \( f(z) \neq z \), then \( z = 0 \) is a zero of order \( m \) of \( f(z) - z \) for some \( m \) with \( m \geq k + 1 \).

**Proof.** We set, for \( z \in B \),
\[
F(z) = f^{-1}\left(e^{-2\pi i/k}f(e^{2\pi i/k}z)\right).
\]
Then \( F \) is biholomorphic from \( B \) into \( B \). Moreover \( F(0) = 0 \) and \( DF(0) = \text{Id} \). It follows from these and the Cartan Theorem that \( F \) is identity, that is,
\[
f(z) = e^{-2\pi i/k}f(e^{2\pi i/k}z).
\]
Therefore \( D^m f(z) = e^{-2(m-1)\pi i/k}D^mf(e^{2\pi i/k}z) \), that is, \( f(0) = 0 \), \( Df(0) = \text{Id} \), \( D^2 f(0) = 0 \), \ldots , \( D^k f(0) = 0 \). This completes the proof.

One of the main results of this section is a growth theorem for mappings in \( S^0_{k+1}(B) \). To this end, we need to use the following lemma (cf. [Ko]).

**Lemma 3.2.** Let \( f \) and \( h \) satisfy the assumptions of Theorem 2.3. Let \( v = v(z,s,t) \) is the solution of the initial value problem (2.1). If \( e^{-t}f(z,t) - z \) has a zero of order \( k + 1 \) for each \( t \geq 0 \), then
\[
||z||\frac{1 - ||z||^k}{1 + ||z||^k} \leq \Re \varphi(h(z,t)) \leq ||z||\frac{1 + ||z||^k}{1 - ||z||^k}
\]
for \( z \in B \), \( \varphi \in T(z) \) and \( t \geq 0 \).

**Proof.** We take a point \( z_0 \) with \( ||z_0|| = 1 \) and set \( g(\zeta) = (1 + \zeta)/(1 - \zeta) \) for \( \zeta \in \Delta \). Let \( p(\cdot,t) : \Delta \to \mathbb{C} \) be given by
\[
p(\xi,t) = \begin{cases} \frac{1}{\xi}\varphi(h(\xi z_0,t)), & \xi \neq 0 \\ 1, & \xi = 0 \end{cases}
\]
for \( \varphi \in T(z_0) \) and \( \zeta \in \Delta \). Then \( p(\cdot, t) \in H(\Delta) \), \( p(0, t) = g(0) = 1 \). Since \( e^{-t} f(z, t) - z \) has a zero of order \( k + 1 \) for each \( t \geq 0 \), for each \( t \geq 0 \), there exists a holomorphic mapping \( F(z, t) \) on a neighbourhood of 0 such that

\[
f(\xi z_0, t) - e^t \xi z_0 = \xi^{k+1} F(\xi z_0, t).
\]

Then

\[
\frac{\partial f}{\partial t}(\xi z_0, t) - e^t \xi z_0 = \xi^{k+1} \frac{\partial F}{\partial t}(\xi z_0, t).
\]

Therefore, we obtain that

\[
p(\xi, t) = \varphi \left( [DF(\xi z_0, t)]^{-1} e^t \xi z_0 \right) + \varphi \left( [DF(\xi z_0, t)]^{-1} \frac{\partial F}{\partial t}(\xi z_0, t) \right) \xi^k.
\]

Thus, there exists a holomorphic function \( \tilde{p}(\xi, t) \) on a neighbourhood of 0 such that \( p(\xi, t) = 1 + \xi^k \tilde{p}(\xi, t) \). Since \( h(z, t) \in \mathcal{M} \), we deduce that \( p(\xi, t) \in g(\Delta) \) for \( \xi \in \Delta \). Therefore, \( g^{-1} \circ p(\cdot, t) : \Delta \to \Delta \) and \( g^{-1} \circ p(0, t) = 0 \). Since \( g^{-1}(1) = 0 \), there exists a holomorphic function \( G(w) \) on a neighbourhood of 1 such that \( g^{-1}(w) = (w - 1)G(w) \). Therefore, we obtain that \( g^{-1} \circ p(\xi, t) = \xi^k \tilde{p}(\xi, t)G(p(\xi, t)) \) on a neighbourhood of 0. Then, by the Schwarz lemma, we obtain that \( |g^{-1} \circ p(\xi, t)| \leq |\xi|^k \) for \( \xi \in \Delta \). Thus, there exists a holomorphic function \( H(\xi) \) on \( \Delta \) such that \( g^{-1} \circ p(\xi, t) = \xi^k H(\xi, t) \) on \( \Delta \) and \( |H(\xi, t)| \leq 1 \) on \( \Delta \). Then, \( p(\xi, t) = g(\xi^k H(\xi, t)) \). Next, in view of the maximum and minimum principle of harmonic functions, we conclude that

\[
\min \{g(|\xi|^k), g(-|\xi|^k)\} \leq \Re p(\xi, t) \leq \max \{g(|\xi|^k), g(-|\xi|^k)\}, \xi \in \Delta.
\]

For \( \xi = \|z\| \) in the above relations, we obtain the inequality (4.1). This completes the proof.

**Lemma 3.3.** Let \( h \) satisfy the assumptions of Theorem 2.3. Let \( v = v(z, s, t) \) be the solution of the initial value problem (2.1). If \( e^{-t} f(z, t) - z \) has a zero of order \( k + 1 \) for each \( t \geq 0 \), then

\[
e^s \|z\| \exp \left( \int_{\|v(z, s, t)\|} \frac{-2x^{k-1}}{1 + x^k} dx \right) \leq e^s \|v(z, s, t)\| \leq e^s \|z\| \exp \left( \int_{\|v(z, s, t)\|} \frac{2x^{k-1}}{1 - x^k} dx \right), \quad (3.2)
\]

for \( z \in B \) and \( t \geq s \geq 0 \).

**Proof.** Fix \( s \geq 0 \) and \( z \in B \setminus \{0\} \) and let \( v(t) = v(z, s, t) \). Also let \( l_z \in T(z) \). Then for all \( t, t_0 \) with \( s \leq t < t_0 \), we have

\[
\left\| v(t) - v(t_0) \right\| \leq \left\| v(t) - v(t_0) \right\| \leq \left\| \int_{t}^{t_0} \frac{dv(\tau)}{d\tau} d\tau \right\| \leq \int_{t}^{t_0} \left\| \frac{dv(\tau)}{d\tau} \right\| d\tau = \int_{t}^{t_0} \| - h(v(\tau), \tau) \| d\tau \leq M(t_0 - t).
\]

Hence \( \|v(t)\| \) is absolutely continuous for \( t \in [s, \infty) \) and thus \( \|v(t)\| \) is differentiable a.e. on \( [s, \infty) \). Moreover,

\[
\frac{\partial \|v\|}{\partial t} = \Re \left[ \varphi \left( \frac{\partial v}{\partial t} \right) \right]
\]

for \( \varphi \in T(v(t)) \) a.e. on \([s, \infty)\), by [Ka, Lemma 1.3]. Equivalently,
\[
\frac{\partial \|v\|}{\partial t} = - \Re \left[ \varphi(h(v,t)) \right], \quad \text{almost everywhere on } [s, \infty).
\] (3.3)

We now integrate both sides of (2.3) with respect to \( t \) to obtain
\[
- \int_{\|z\|}^{\|v\|} \frac{1 + x^k}{x(1 - x^k)} \, dx = - \int_{s}^{t} \frac{1 + \|v(\tau)\|^k}{\|v(\tau)\||(1 - \|v(\tau)\|^k)} \cdot \frac{d\|v(\tau)\|}{d\tau} \, d\tau \geq \int_{s}^{t} d\tau = t - s
\]
and
\[
- \int_{\|z\|}^{\|v\|} \frac{1 - x^k}{x(1 + x^k)} \, dx = - \int_{s}^{t} \frac{1 - \|v(\tau)\|^k}{\|v(\tau)\||(1 + \|v(\tau)\|^k)} \cdot \frac{d\|v(\tau)\|}{d\tau} \, d\tau \geq \int_{s}^{t} d\tau = t - s.
\]

Finally straightforward computations in the above relations yield (2.2), as desired. This completes the proof.

We now are able to obtain the following growth result for the set \( S_{k+1}^0(B) \).

**Theorem 3.4.** If \( f \in S_{k+1}^0(B) \), then
\[
\frac{\|x\|}{(1 + \|x\|^k)^{\frac{k}{k-1}}} \leq \|f(x)\| \leq \frac{\|x\|}{(1 - \|x\|^k)^{\frac{k}{k-1}}} \quad \text{for all } x \in B.
\] (3.4)

**Proof.** Since \( f \in S_{k+1}^0(B) \) we have
\[
f(z) = \lim_{t \to \infty} e^t v(z, t)
\] (3.5)
locally uniformly on \( B \), where \( v = v(z, t) \) is the solution of the initial value problem
\[
\frac{\partial v}{\partial t} = -h(v, t), \quad \text{a.e. } t \geq 0, \quad v(z, 0) = z,
\]
for all \( z \in B \). Taking into account the relations (2.1), one deduces that
\[
\|z\| \exp \int_{\|v(z, t)\|}^{\|x\|} \frac{-2x^{k-1}}{1 + x^k} \, dx \leq e^t \|v(z, t)\| \leq \|z\| \exp \int_{\|v(z, t)\|}^{\|x\|} \frac{2x^{k-1}}{1 - x^k} \, dx,
\] (3.6)
for \( z \in B, \quad t \geq 0 \).

Since
\[
\lim_{t \to \infty} e^t \|v(z, t)\| = \|f(z)\| < \infty,
\]
we must have
\[
\lim_{t \to \infty} \|v(z, t)\| = \lim_{t \to \infty} e^{-t} \|e^t v(z, t)\| = 0.
\]

Letting \( t \to \infty \) in (3.6) and using (3.5), we obtain the estimate (3.4), as desired. This completes the proof.

We remark that if \( f(z, t) \) is a Loewner chain, then using similar reasoning as in the above result, we obtain the following growth theorem.
Corollary 3.5. Let \( f(z,t) \) be a Loewner chain. If \( e^{-t}f(z,t) - z \) has a zero of order \( k+1 \) for each \( t \geq 0 \), then

\[
\frac{\|x\|}{(1 + \|x\|^k)^{\frac{1}{k}}} \leq \|e^{-t}f(z,t)\| \leq \frac{\|x\|}{(1 - \|x\|^k)^{\frac{1}{k}}}
\]

for \( z \in B, \ t \geq 0 \).

REFERENCES


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