RIGIDITY OF BERGMAN LENGTH ON RIEMANN SURFACES
UNDER PSEUDOCONVEXITY

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ABSTRACT. We study the rigidity of the Bergman metrics on Riemann surfaces under pseudoconvexity. Then we apply it to show the rigidity of the Bergman length of an arc on Riemann surfaces under pseudoconvexity.

1. INTRODUCTION

Let \( \pi : \mathcal{R} \to B \) be a holomorphic family of Riemann surfaces \( R(t) = \pi^{-1}(t), t \in B \), where \( \mathcal{R} \) is a complex 2-dimensional manifold and \( B = \{ |t| < \rho \} \) is a disk in \( \mathbb{C}_t \). We consider a subdomain \( \mathcal{D} \) of \( \mathcal{R} \) such that the restriction \( \pi : \mathcal{D} \to B \) induces a holomorphic family of Riemann surfaces \( D(t) = \pi^{-1}(t), t \in B \) with the following conditions:

1. \( D(t) \subset \subset R(t), t \in B \);
2. \( \partial D(t) \) consists of a finite number of \( C^\infty \) smooth closed curves;
3. \( \partial D(t) \) varies \( C^\infty \) smoothly with parameter \( t \in B \).

Each \( D(t), t \in B \) carries the Bergman metric \( ds_t^2 := K(t, z)|dz|^2 \). Then in [2] based on [4], Maitani-Yamaguchi gave the variation formulas for the Bergman metrics (Theorem 2 mentioned below), and applied them to show the following property which is variation of the Bergman metrics on \( D(t) \) under pseudoconvexity.

Lemma 1 ([2]). If \( \mathcal{D} \) is pseudoconvex in \( \mathcal{R} \), then log \( K(t, z) \) is plurisubharmonic on \( \mathcal{D} \).

From now on, we always assume that \( \mathcal{D} \) is pseudoconvex in \( \mathcal{R} \). In [1], we apply this lemma to get:

Theorem 1. If there exists a holomorphic section \( \sigma : t \in B \mapsto \zeta(t) \in D(t) \) on \( B \) of \( \mathcal{D} \) such that log \( K(t, \zeta(t)) \) is harmonic on \( B \), then \( \mathcal{D} \cong B \times D(0) \), that is, \( \mathcal{D} \) is biholomorphic to the product \( B \times D(0) \) by the transformation of the form \( t = t, w = f(t, z) \).

In this paper, we shall show the rigidity of Bergman length of an arc on Riemann surfaces as an application of Theorem 1. Let \( \gamma \) be a smooth arc from \( a \) to \( b \) in \( \mathbb{C}_z \). Assume that there exists a neighborhood \( V \) of \( \gamma \) in \( \mathbb{C}_z \) and a holomorphic mapping \( T : (t, z) \in B \times V \mapsto (t, \varphi(t, z)) \in \mathcal{D} \). For each \( t \in B \), we put \( \gamma(t) = \varphi(t, \gamma) \) which is an arc in \( D(t) \). If we consider the Bergman length \( l(t) \) of \( \gamma(t) \) on \( D(t) \), that is,

\[
l(t) := \int_{\gamma(t)} ds_t(z) = \int_{\gamma(t)} \sqrt{K(t, z)}|dz|.
\]
Then we have:

**Corollary 1.** Assume that $D$ is pseudoconvex in $\mathcal{R}$. Then the following holds:

(i) $\log I(t)$ is a subharmonic function on $B$.

(ii) If there exists at least one arc $\gamma$ defined above such that $\log I(t)$ is a harmonic function on $B$, then $D \cong B \times D(0)$.

2. Variation of Bergman metrics

We may assume from Nishimura’s theorem [3] that $D$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary such that each boundary $\partial D(t), t \in B$ of $D(t)$ is also smooth.

To prove Theorem 1, we first recall the canonical function $Q(t, z, \zeta)$ introduced in [2]. Let $D$ be a domain over $B \times \mathbb{C}_z$ with smooth boundary. Let $t \in B$ and $\zeta \in D(t)$ be fixed. We consider the complex-valued harmonic function $\psi(t, z, \zeta)$ on $D(t) - \zeta$ with pole at $\zeta$ such that $\psi(t, z, \zeta) + \frac{1}{i} \frac{1}{\pi} \frac{1}{z - \zeta}$ is harmonic on a neighborhood of $\zeta$ in $D(t)$ and $\psi(t, z, \zeta)$ vanishes continuously on $\partial D(t)$. We make the differential $d_z \psi(t, z, \zeta)$ with respect to $z$ and decompose it into

$$d_z \psi(t, z, \zeta) = \partial \psi(t, z, \zeta) + \bar{\partial} \psi(t, z, \zeta) \equiv \mathcal{L}(t, z, \zeta) dz + \mathcal{K}(t, z, \zeta) d\zeta,$$

where $\mathcal{L}(t, z, \zeta) = \frac{\partial \psi(t, z, \zeta)}{\partial z}$, $\mathcal{K}(t, z, \zeta) = \frac{\bar{\partial} \psi(t, z, \zeta)}{\partial z}$. Thus, $\mathcal{L}(t, z, \zeta)$ is a meromorphic function on $D(t)$ with pole $\frac{1}{\pi} \frac{1}{(z - \zeta)^2}$ at $\zeta$, and $\mathcal{K}(t, z, \zeta)$ is a holomorphic function on $D(t)$. Then $\mathcal{K}(t, z, \zeta) dz d\zeta$ is identical with the Bergman invariant on $D(t)$, that is, $f(\zeta) = \int_{D(t)} f(z) \mathcal{K}(t, z, \zeta) dz dy$ for every $L^2$ holomorphic function $f(z)$ on $D(t)$ (see [4] and cf. [5]). We set

$$Q(t, z, \zeta) = \mathcal{K}(t, z, \zeta) \mathcal{L}(t, z, \zeta), \quad z \in D(t).$$

From $\mathcal{K}(t, \zeta, \zeta) = K(t, \zeta) > 0$, $Q(t, z, \zeta)$ is a meromorphic function for $z \in D(t)$ with zero of order 2 at $\zeta$. Since $\psi(t, z, \zeta) = 0$ on $\partial D(t)$, $|Q(t, z, \zeta)| = 1$ on $\partial D(t)$. We called $Q(t, z, \zeta)$ the canonical function for $(D(t), \zeta)$.

Under this notation, we recall the following variation formula of the second order for the coefficient of the Bergman metric $K(t, \zeta)|d\zeta|^2$ on $D(t)$.

**Theorem 2** ([2]). It holds for $(t, z) \in D$

$$\frac{\partial^2 K(t, \zeta)}{\partial t \partial \bar{t}} = \frac{1}{4} \int_{\partial D(t)} k_2(t, z) \left(|\mathcal{L}(t, z, \zeta)|^2 + |\mathcal{K}(t, z, \zeta)|^2 \right) ds_z$$

$$+ \int \int_{D(t)} \left(\left| \frac{\partial \mathcal{L}(t, z, \zeta)}{\partial \bar{t}} \right|^2 + \left| \frac{\partial \mathcal{K}(t, z, \zeta)}{\partial \bar{t}} \right|^2 \right) dx dy,$$

where

$$k_2(t, z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \frac{\partial \varphi}{\partial z} \right)^2 - 2 \Re \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{z}} \frac{\partial \varphi}{\partial \zeta} \right\} + \left| \frac{\partial \varphi}{\partial t} \frac{\partial^2 \varphi}{\partial \bar{z} \partial \bar{z}} \right|^2 \left| \frac{\partial \varphi}{\partial \zeta} \right|^3$$

for $(t, z) \in \partial D$. Here $\varphi(t, z)$ is a defining function of $\partial D$ in $\mathcal{R}$.

We remark that $k_2(t, z)$ on $\partial D$ does not depend on the choice of defining functions $\varphi(t, z)$ of $\partial D$. 
3. Rigidity of Bergman metrics

To prove Theorem 1, we next prepare the following two lemmas. Let $B \times V = \{ |t| < \rho \} \times \{ |z| < r \} \subset \mathbb{C}^2$. Let $l$ be $C^\omega$ smooth arc in $\mathbb{C}_z$ such that $l$ divides $V$ into two parts $V^+$ and $V^-$. Let $u(t, z), t \in B$ be a harmonic function on $V^+$. Then we have

**Lemma 2.** If each $u(t, z), t \in B$ vanishes on $(\partial V^+) \cap l$ and $\frac{\partial u}{\partial z}(t, z)$ is holomorphic for two complex variables $(t, z)$ in $B \times V^+$, then $u(t, z)$ does not depend on $t \in B$.

Let $C : |z| = 1$ be a circle. For each $t \in B$, let $L(t)$ be a subarc in $C$ (closed as set) and set $\mathcal{L} = \bigcup_{t \in B}(t, L(t)) \subset B \times \mathbb{C}_z$. Then we have

**Lemma 3.** Assume that $\mathcal{L}$ is pseudoconcave in $B \times \mathbb{C}_z$, i.e., $(B \times \mathbb{C}_z) - \mathcal{L}$ is pseudoconvex. Then $L(t)$ does not depend on $t \in B$, that is, $\mathcal{L} = B \times L(0)$.

The outline of the proof of Theorem 1 is as follows: we may first assume that the holomorphic section on $B$ of $\mathcal{D}$ is constant $z = 0$ on $B$ and $K(t, 0) \equiv 1$ on $B$. Then we see that the canonical function $Q(t, z, 0)$ for $(D(t), 0), t \in B$ is a meromorphic function for $(t, z)$ on $\mathcal{D}$. In fact, we apply Theorem 1 to $K(t, 0) \equiv 1$ on $B$ and $k_2(t, z) \geq 0$ on $\partial D$ since $\mathcal{D}$ is pseudoconvex. Then we find $\mathcal{L}(t, z, 0)$ and $K(t, z, 0)$ are holomorphic functions for $t$ in $B$. Therefore, $\mathcal{K}$ and $\mathcal{L}$ are holomorphic functions for $(t, z)$ in $\mathcal{D}$ and $\mathcal{D} - (B \times \{0\})$, respectively. Thus, $Q(t, z, 0)$ is a meromorphic function for $(t, z)$ in $\mathcal{D}$. Finally, by Lemma 2 and 3, we can prove that the analytic transformation $S : t = t, w = Q(t, z, 0)$ bijectively maps $\mathcal{D}$ onto the product $B \times S(0)$, where $S(0) = Q(0, D(0), 0)$.

4. Rigidity of Bergman lengths

We can apply Theorem 1 to show the rigidity of Bergman length of an arc on Riemann surfaces under the condition that $\mathcal{D}$ is pseudoconvex. Here we shall prove Corollary 1.

**Proof of Corollary 1 (i).** Since $T : (t, z) \in B \times V \rightarrow (t, \varphi(t, z)) \in \mathcal{D}$ is a holomorphic mapping and since each Bergman metric $K(t, z)[dz]^2, t \in B$ is invariant under holomorphic mappings for $z$, it follows that we may assume that $\gamma$ is contained in each $D(t), t \in B$, namely, $\gamma$ is independent of $t \in B$.

From the definition of integral, $l(t)$ can be written in the form:

$$l(t) = \lim_{n \to \infty} \frac{L}{n} \sum_{i=1}^{n} \sqrt{K(t, z_i)},$$

where $L$ is the Euclidean length of $\gamma$ and $\{z_i\}_{i=1, \ldots, n}$ are the partition of $\gamma$ with same length. We note that the above limit $l(t)$ is uniform convergence for $t \in B$. It thus suffices to prove that, for each $n = 1, 2, \ldots$, $\log(\sum_{i=1}^{n} \sqrt{K(t, z_i)})$ is subharmonic on $B$. In fact, by a direct calculation, we have the following formula:

$$\frac{\partial^2}{\partial t \partial \bar{t}} \left( \log \sum_{i=1}^{n} e^{s_i} \right) = \left( \sum_{i=1}^{n} e^{s_i} \right)^{-2} \left( \sum_{i=1}^{n} e^{s_i} \sum_{i=1}^{n} e^{s_i} \frac{\partial^2 s_i}{\partial t \partial \bar{t}} + \sum_{i \neq j} e^{s_i+s_j} \left| \frac{\partial s_i}{\partial t} - \frac{\partial s_j}{\partial \bar{t}} \right|^2 \right).$$

Here $s_i(t) = \log \sqrt{K(t, z_i)}$ for $i = 1, \ldots, n$. Since each $s_i$ is subharmonic on $B$ by Lemma 1, the right-hand side of the above equation is non-negative, and hence $\log(\sum_{i=1}^{n} e^{s_i})$ is subharmonic. Therefore, we conclude that $\log l(t)$ is subharmonic on $B$. \qed
Proof of Corollary 1 (ii). We fix a point $c \in \gamma$, $c \neq a, b$. Then we shall show that, under the condition of (ii): $\log l(t)$ is harmonic on $B$, both $\log l_1(t)$ and $\log l_2(t)$ are harmonic on $B$. Here

$$l_1(t) := \int_a^c \sqrt{K(t,z)}|dz| \quad \text{and} \quad l_2(t) := \int_c^b \sqrt{K(t,z)}|dz|.$$  

In fact, if we set $u_i(t) = \log l_i(t)$ for $i = 1, 2$, then we have $l(t) = l_1(t) + l_2(t) = e^{u_1(t)} + e^{u_2(t)}$. It follows from the assumption that

$$\frac{\partial^2}{\partial t \partial \bar{t}} \log(e^{u_1(t)} + e^{u_2(t)}) = 0$$

$$\iff \frac{e^{u_1(t)} + e^{u_2(t)}}{(e^{u_1(t)} + e^{u_2(t)})^2} \left( \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \right)^2 + \frac{1}{e^{u_1(t)} + e^{u_2(t)}} \left( e^{u_1(t)} \frac{\partial^2 u_1}{\partial t \partial \bar{t}} + e^{u_2(t)} \frac{\partial^2 u_2}{\partial t \partial \bar{t}} \right) = 0.$$  

Since each $u_i$ is subharmonic on $B$ by Corollary 1(i), we have

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t}, \quad \frac{\partial^2 u_1}{\partial t \partial \bar{t}} = \frac{\partial^2 u_2}{\partial t \partial \bar{t}} = 0.$$  

Thus $u_i$ must be harmonic on $B$.

For an arbitrary fixed $\epsilon$, let $c = a + \epsilon$. From the above, we see that $\log \int_a^{a+\epsilon} \sqrt{K(t,z)}|dz|$ is harmonic on $B$, and hence $\log \frac{1}{\epsilon} \int_a^{a+\epsilon} \sqrt{K(t,z)}|dz|$ is also harmonic on $B$. From the definition of integral,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_a^{a+\epsilon} \sqrt{K(t,z)}|dz| = \log \sqrt{K(t,a)}.$$  

Therefore, $\log \sqrt{K(t,a)}$ is harmonic on $B$. By Theorem 1, we conclude $D \cong B \times D(0)$.  

References


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