THE UNIQUENESS THEOREMS OF ALGEBROID FUNCTION

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ABSTRACT. First we define the operation of addition and multiplication. As an applying of the new definition of addition, we extend the uniqueness theorem for meromorphic function by C. C. Yang and H. X. Yi to the algebroid function with multiple values.

1. INTRODUCTION AND MAIN RESULTS

Suppose that $A_k(z),...,A_0(z)$ are analytic functions defined on the complex plane $\mathbb{C}$ without common zeros, then the following equation

$$\Psi(z, W) = A_k(z)W^k + A_{k-1}(z)W^{k-1} + ... + A_0(z) = 0$$

(1)

defines a $k$-valued algebroid function $W(z)$ on $\mathbb{C}$. If $\Psi(z, W)$ is irreducible, we call $W(z)$ a $k$-value irreducible algebroid function. A general $k$-value algebroid function $W(z)$ may split up into several ($n$) irreducible algebroid functions (containing $W = c$, $c$ is a constant), but the sum of the new value numbers $k_1$, $k_2$, ..., $k_n$ caused by the split is equal to the initial number $k = \sum_{j=1}^{n} k_j$. In this paper, we generally do not require $\Psi(z, W)$ is irreducible. When $k = 1$, then $W(z)$ is a meromorphic function. In this paper, we extend the uniqueness theorem for meromorphic function by C. C. Yang and H. X. Yi (see e.g. [10]) to the algebroid function with multiple values. In order to introduce the main results, we now give the following definitions.

**Definition 1.** We call $b \in \mathbb{C}_\infty$ a $t$-common value of two algebroid functions $W(z)$ and $M(z)$ or a common value for short, if $\overline{E}_t(W = a) = \overline{E}_t(M = a)$. Here $\overline{E}_t(a_j, f)$ denotes the set of zeros of $f - a_j$ with multiplicities no greater than $k$, in which each zero counted only once.

**Definition 2.** Assume that $w(z)$ is a $\nu$ valued algebroid function, $a \in \mathbb{C}_\infty$. Set $\delta(a, w) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{w - a}{T(r, w)})}{T(r, w)}$, and $\Delta(a, w) = 1 - \liminf_{r \to \infty} \frac{N(r, \frac{w - a}{T(r, w)})}{T(r, w)}$. We call $\delta(a, w)$ the Nevanlinna deficiency of the value $a$, and $\Delta(a, w)$ the Valiron deficiency of the value $a$.

**Definition 3.** Let $W(z)$ be an algebroid function defined by $\Psi(z, W)$, $b \in \overline{C}$ a finite or an infinite complex number. We will denote the inverse image of $b$ by

$$W^{-1}(b) = \{z_0 \in C; \Psi(z_0, b) = 0\}.$$

**Definition 4.** We call $b \in \overline{C}$ the common value of two algebroid function $W(z)$ and $M(z)$; that is to say, $W^{-1}(b) = M^{-1}(b)$.

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A classical result about uniqueness for meromorphic function is the five values theorem due to Nevanlinna in 1929. After that, there have been many papers on the uniqueness for meromorphic function. We refer the reader to the book [10]. In the first section of this paper, we extend C. C. Yang’s strengthening theorem (see e.g. [10]) of meromorphic functions to algebroid functions and prove the following theorem.

**Theorem 1.** Let \( W(z) \) be a \( k \)-valued irreducible algebroid function and \( M(z) \) be an \( s \)-valued algebroid function. The elements of \( B := \{b_j\}_{j=1}^{p} \) are \( p = 2k + 2s + 1 \) distinct finite or infinite complex numbers. If for any \( b \in B: W^{-1}(b) \subset M^{-1}(b) \), and

\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{p} N(r, W - b_j = 0)}{\sum_{j=1}^{p} N(r, M - b_j = 0)} > \frac{s(2s + 1)}{(2k + 1)(s + 1)}.
\]

Then \( W(z) \equiv M(z) \).

From Theorem 1, we can obtain the following results.

**Corollary 1.** (Nevanlinna five value theorem) Let irreducible \( k \)-valued algebroid function \( W(z) \) and irreducible \( s \)-valued algebroid function \( M(z) \) have \( p = 4k + 1 \) distinct common values \( b_j, (j = 1, 2, \ldots, p) \) (finite or infinite), then \( W(z) \equiv M(z) \).

**Corollary 2.** Let \( W(z) \) and \( M(z) \) be two irreducible \( k \)-valued algebroid function; the elements of \( B := \{b_j\}_{j=1}^{p} \) be \( p = 4k + 1 \) distinct finite or infinite complex numbers. If for any \( b \in B: W^{-1}(b) \subset M^{-1}(b) \), and \( \liminf_{r \to \infty} \frac{\sum_{j=1}^{p} N(r, W - b_j = 0)}{\sum_{j=1}^{p} N(r, M - b_j = 0)} > \frac{k}{k+1} \), Then \( W(z) \equiv M(z) \).

When there exist deficient values, H. X. Yi (see e.g. [10]) improved a H. Ueda’s [8] uniqueness theorem for meromorphic functions and obtained the following theorems.

**Theorem A.** Let \( f_1(z) \) and \( f_2(z) \) be two non-constant meromorphic functions, \( a_j (j = 1, 2, \ldots, q) \) be \( q \) distinct complex numbers, and let \( t_j \in \mathbb{N} \cup \{\infty\} \) satisfy

\[
1 \leq t_q \leq t_{q-1} \leq \ldots \leq t_2 \leq t_1 \leq \infty,
\]

as well as

\[
E_{t_j}(a_j, f_1(z)) = E_{t_j}(a_j, f_2(z)), \quad j = 1, 2, \ldots, q.
\]

Set

\[
A_i = \frac{\delta(a_1, f_i) + \delta(a_2, f_i)}{t_3 + 1} + \sum_{j=3}^{q} \frac{\delta(a_j, f_i)}{t_j + 1}, \quad i = 1, 2.
\]

If \( \min\{A_1, A_2\} \geq 2 - \sum_{j=3}^{q} \frac{t_j}{t_j + 1} \) and \( \max\{A_1, A_2\} > 2 - \sum_{j=3}^{q} \frac{t_j}{t_j + 1} \), then \( f_1(z) \equiv f_2(z) \).

**Theorem B.** Let \( f_1(z) \) and \( f_2(z) \) be two non-constant meromorphic functions of finite order, \( a_j (j = 1, 2, \ldots, q) \) be \( q \) distinct complex numbers, and let \( t_j \in \mathbb{N} \cup \{\infty\} \) satisfy (2) and (3). Set

\[
B_i = \frac{\Delta(a_1, f_i) + \Delta(a_2, f_i)}{t_3 + 1} + \sum_{j=3}^{q} \frac{\Delta(a_j, f_i)}{t_j + 1}, \quad i = 1, 2.
\]

If \( \sum_{j=3}^{q} \frac{t_j}{t_j + 1} = 2 \) and \( \max\{B_1, B_2\} > 0 \), then \( f_1(z) \equiv f_2(z) \).
In this paper, referring to the method for meromorphic functions dealing with multiple values due to L. Yang, we extend Theorem A and Theorem B for algebroid functions and prove the following theorems.

**Theorem 2.** Assume that \( W_1(z) \) and \( W_2(z) \) are \( k \) valued irreducible algebroid functions, \( a_j(j = 1, 2, \cdots, q) \) be \( q \) distinct complex numbers, and let \( t_j \in \mathbb{N} \cup \{ \infty \} \) satisfy (2) and
\[
E_{t_j}(a_j, W_1) = E_{t_j}(a_j, W_2), \quad j = 1, 2, \cdots, q.
\]
Set
\[
A_i = \frac{\delta(a_1, W_1) + \delta(a_2, W_1)}{k_{2i+1}} + \sum_{j=2k+1}^{q} \frac{\delta(a_j, W_1)}{k_{j+1}}, \quad i = 1, 2. \quad \text{If}
\]
\[
\min\{A_1, A_2\} \geq 2k - \sum_{j=2k+1}^{q} \frac{t_j}{t_j + 1}, \quad \text{and} \quad \max\{A_1, A_2\} > 2k - \sum_{j=2k+1}^{q} \frac{t_j}{t_j + 1},
\]
then \( W_1(z) \equiv W_2(z) \).

**Theorem 3.** Let \( W_1(z) \) and \( W_2(z) \) be two \( k \) valued irreducible algebroid functions of finite order, \( a_j(j = 1, 2, \cdots, q) \) be \( q \) distinct complex numbers. Assume that \( t_j \in \mathbb{N} \cup \{ \infty \} \) satisfy (2) and (4), and \( B_i = \frac{\Delta(a_1, W_1) + \Delta(a_2, W_1)}{k_{2i+1}} + \sum_{j=2k+1}^{q} \frac{\Delta(a_j, W_1)}{k_{j+1}}, \quad i = 1, 2. \) If
\[
\sum_{j=2k+1}^{q} \frac{t_j}{t_j + 1} = 2k, \quad \text{and} \quad \max\{B_1, B_2\} > 0.
\]
then \( W_1(z) \equiv W_2(z) \).

2. **Some Lemmas**

Let \( w(z) \) be a \( \nu \) valued algebroid function defined by (1). By [7], we know that \( w(z) \) can be decomposed into \( \nu \) one-valued branches of functions \( w_j, j = 1, 2, \cdots, \nu \) meromorphic in the domain \( T_w = \mathbb{C} - H \), where \( H \) is an acyclic polyline which links all the critical points of \( w(z) \). Suppose that \( u_i \) and \( u_0 \) are one-valued branches of two algebroid (\( m \) valued and \( n \) valued) functions. It follows from Prokopovich [4], we consider their sum (difference, product, quotient) in the domain \( T_{w,u} \) of the complex plane with a cutting through the projection of the critical points of both functions. The function \( w + u(w - u, wu, w/u) \) is an \( mn \) valued algebroid function, the one-valued branches of which will be defined \( w_j + u_i(w_j - u_i, w_ju_i, w_j/u_i) \) where \( 1 \leq j \leq m, 1 \leq i \leq n \). Recently, Sun and Gao [6] give a detailed discussion of the definition of sum (difference, product, quotient).

For the sake of convenience, we also introduce the following notations.

Suppose that \( w = w(z) \) is a \( \nu \) valued algebroid function defined by (1) on the complex plane. Selberg [5] introduced the following Nevanlinna quantites for an algebroid function
\[
N(r, w) = \frac{1}{\nu} \int_0^r \frac{n(t, w) - n(0, w)}{t} dt + \frac{n(0, w)}{\nu} \log r,
\]
\[
m(r, w) = \frac{1}{2\pi\nu} \sum_{j=1}^{\nu} \int_0^{2\pi} \log^+ |w_j(re^{i\theta})| d\theta, \quad z = re^{i\theta},
\]
\[
T(r, w) = m(r, w) + N(r, w),
\]
where \( \nu \) is the number of branches of \( w(z) \) and proved for them the fundamental propositions of Nevanlinna theory, in particular, the first fundamental theorem and the lemma on the logarithmic derivative (see [4]). Although Selberg considered only irreducible algebroid function, he did not use the irreducibility of \( \Phi(z, w) \) in (1). Therefore, the
first fundamental theorem is also true for algebroid function in our terminology. Denote \( \overline{n}_{k}(r, w = a) \) by the zeros of \( w(z) - a \) in \( |z| \leq r \) with multiplicities not greater than \( k \), and \( \overline{n}_{k+1}(r, w = a) \) by the zeros of \( w(z) - a \) in \( |z| \leq r \) with multiplicities not less than \( k + 1 \). Similarly, we can define \( \overline{N}_{k}(r, W = a) \) and \( \overline{N}_{k+1}(r, W = a) \) as Selberg [5].

In [6], Sun and Gao have proved the following lemmas which are useful in the proof of our main results.

**Lemma 1.** Assume that \( W(z) \) is \( k \) valued algebroid function and \( M(z) \) is \( s \) valued algebroid function, the sets of \( W(0) \) and \( M(0) \) have no poles, then

\[
T(r, W \pm M) \leq T(r, W) + T(r, M) + \log 2; \quad T(r, W \cdot M) \leq T(r, W) + T(r, M).
\]

**Lemma 2.** Assume that \( W(z) \) is a \( v \) valued irreducible algebroid function, \( \{a_{j}\}_{j=1}^{q} \subset \mathbb{C}_{\infty} \) be \( q \) distinct complex numbers, and \( k_{j}, (j = 1, 2, \cdots, q) \) be \( q \) positive integers, then

\[
(q - 2v)T(r, W) < \sum_{j=1}^{q} \frac{k_{j}}{k_{j} + 1} \overline{N}_{k_{j}}(r, W = a_{j}) + \sum_{j=1}^{q} \frac{1}{k_{j} + 1} N(r, W = a_{j}) + S(r, W);
\]

\[
(q - 2v - \sum_{j=1}^{q} \frac{1}{k_{j} + 1})T(r, W) < \sum_{j=1}^{q} \frac{k_{j}}{k_{j} + 1} \overline{N}_{k_{j}}(r, W = a_{j}) + S(r, W).
\]

### 3. Proof of Theorems

We only prove Theorem 2 by using Lemma 1 and Lemma 2, since we can obtain Theorem 1 and Theorem 3 by the similar method of the proof. Using Lemma 1 we can obtain Theorem 1 by the similar method of the proof as that in [10].

**Proof of Theorem 2** We distinguish two cases below.

**Case 1.** All the values \( \{a_{j}\} \) are finite. Suppose that \( W_{1}(z) \neq W_{2}(z) \). Since

\[
\sum_{j=1}^{q} \overline{n}_{t_{j}}(r, W_{1} = a_{j}) = \sum_{j=1}^{q} \overline{n}_{t_{j}}(r, W_{2} = a_{j}) \leq n(r, W_{1} - W_{2} = 0),
\]

then \( \sum_{j=1}^{q} \overline{N}_{t_{j}}(r, W_{1} = a_{j}) \leq kN(r, W_{1} - W_{2} = 0) \). It follows from the first fundamental theorem and Lemma 1 that

\[
\sum_{j=1}^{q} \overline{N}_{t_{j}}(r, W_{1} = a_{j}) \leq kN(r, W_{1} - W_{2} = 0) \leq kT(r, W_{1} - W_{2}) + O(1) \leq kT(r, W_{1}) + kT(r, W_{2}) + O(1).
\]

We can derive from (2) that \( \frac{1}{t_{q+1}} \leq \frac{t_{q}}{t_{q} - 1} \leq \cdots \leq \frac{t_{2}}{t_{2} + 1} \leq \frac{t_{1}}{t_{1} + 1} \leq 1 \). Without loss of generality, we can conclude that the sets of \( W_{1}(0) \) and \( W_{2}(0) \) have no poles. Otherwise, we substitute them by \( W_{1}(z)z^{n} \) and \( W_{2}(z)z^{n} \). Combining (5), (7) and Lemma 2, we have

\[
(q - 2k)T(r, W_{1}) < \frac{k_{t_{2k+1}+1}}{t_{2k+1}+1} (T(r, W_{1}) + T(r, W_{2})) + \frac{1}{t_{2k+1}+1} \sum_{j=1}^{2k} (1 - \delta(a_{j}, W_{1}) + o(1))T(r, W_{1}) + \sum_{j=2k+1}^{q} \frac{1}{t_{j}+1} (1 - \delta(a_{j}, W_{1}))T(r, W_{1}) + S(r, W_{1}).
\]

Hence

\[
(A_{1} + \frac{k_{t_{2k}+1}}{t_{2k}+1} + \sum_{j=2k+1}^{q} \frac{t_{j}}{t_{j}+1} - 2k - o(1))T(r, W_{1}) < \frac{k_{t_{2k}+1}}{t_{2k}+1} T(r, W_{2}) + S(r, W_{1}).
\]

Combining this with (5) and (8), we obtain

\[
T(r, W_{1}) < T(r, W_{2}) + S(r, W_{1}) < T(r, W_{2}) + S(r, W_{2}).
\]
Similarly,
\[(A_2 + \frac{kt_{2k+1}}{t_{2k+1}+1} + \sum_{j=2k+1}^{q} \frac{t_j}{t_j+1} - 2k - o(1))T(r, W_2) < \frac{kt_{2k+1}}{t_{2k+1}+1} T(r, W_1) + S(r, W_2),\]  
(10)

and
\[T(r, W_2) < T(r, W_1) + S(r, W_2) < T(r, W_1) + S(r, W_1).\]  
(11)

(9), (11) yield
\[(A_1 + \sum_{j=2k+1}^{q} \frac{t_j}{t_j+1} - 2k - o(1))T(r, W_1) < S(r, W_1),\]  
(12)

while (8), (10) mean
\[(A_2 + \sum_{j=2k+1}^{q} \frac{t_j}{t_j+1} - 2k - o(1))T(r, W_2) < S(r, W_2).\]  
(13)

However, according to (5), we know that one of inequality (12) and (13) is a contradiction.

**Case 2.** If \(a_q = \infty\). For \(c \neq a_j, j = 1, 2, \cdots, q - 1\), put \(F(z) = \frac{1}{W_{j-1}}, G(z) = \frac{1}{W_q - c}\).

Set \(b_j = \frac{1}{a_j - c}, j = 1, 2, \cdots, q - 1, \quad b_q = 0,\) then
\[E_{t_j}(b_j, W_1) = E_{t_j}(b_j, W_2), \quad j = 1, 2, \cdots, q.\]

As we did in Case 1, we can see that \(W_1(z) \equiv W_2(z)\), and Theorem 2 follows.

**References**


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