REMARKS ON THE FOURIER-BOREL TRANSFORMATION ON SOME VARIETIES

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ABSTRACT. We have studied the Fourier-Borel transformation for analytic functionals on some balls and the complex sphere, and for entire functionals of exponential type. Recently, A. Kowata and M. Moriwaki considered the Fourier-Borel transformation on some variety. In this paper, we will generalize a theorem of Kowata-Moriwaki using our method in our previous works.

INTRODUCTION

We call a ball defined by a norm $N$ on $\mathbb{C}^n$ an $N$-ball. In 1960's, A. Martineau [3] determined the images of analytic functionals on the $N$-ball and of entire functionals of exponential type under the Fourier-Borel transformation (Theorem 1). In 1980's, the second named author and R. Wada considered the Fourier-Borel transformation on the complex sphere and proved a theorem (Theorem 2) similar to that of A. Martineau. Then in 1990's, we considered the Fourier-Borel transformation on the dual space of eigen functions of the Laplacian and proved a theorem (Theorem 3) similar to that of A. Martineau. Recently, A. Kowata and M. Moriwaki [2] considered the Fourier-Borel transformation on some variety and they gave a necessary and sufficient condition that the Fourier-Borel transform of the space of entire functionals of exponential type on the variety is isomorphic to the space of entire functions which satisfy some differential equation (Theorem 6). In this paper, first we review our previous works on the Fourier-Borel transformation and will generalize our previous theorems (Theorems 4 and 5). Then we introduce the theorem of Kowata-Moriwaki and we will state their theorem in more general context (Theorems 7 and 8) with an outline of our proof following Fujita-Morimoto [1]. For more references on this paper see [1]. Further information on our previous works, see Morimoto [4].

1. Fourier-Borel transformation

Let $N$ be a norm on $\mathbb{C}^n$ and $N^*$ the dual norm of $N$; $N^*(z) = \text{sup}\{|z \cdot w|; N(w) \leq 1\}$, where $z \cdot w = z_1w_1 + \cdots + z_nw_n$ is the canonical inner product for $z = (z_1, \cdots , z_n), w = (w_1, \cdots , w_n) \in \mathbb{C}^n$. Define $N$-balls of radius $r$ by

$$\tilde{B}_N(r) = \{z \in \mathbb{C}^n; N(z) < r\}, \tilde{B}_N[r] = \{z \in \mathbb{C}^n; N(z) \leq r\}.$$

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Note that $\tilde{B}_N(\infty) = C^n$. We denote by $O(\tilde{B}_N(r))$ the space of holomorphic functions on $\tilde{B}_N(r)$ equipped with the topology of uniform convergence on compact sets and by $O(\tilde{B}_N[r])$ the space of germs of holomorphic functions on $\tilde{B}_N[r]$. For an analytic set $X \subset C^n$ or $X = C^n$, we will consider the spaces of entire functions of exponential type:

$$\exp(X; [r, N]) = \left\{ f \in O(X); \sup_{z \in X} \{|f(z)| \exp(-r'N(z))\} < \infty, \text{ for some } r' < r \right\},$$

$$\exp(X; (r, N)) = \left\{ f \in O(X); \sup_{z \in X} \{|f(z)| \exp(-r'N(z))\} < \infty, \text{ for any } r' > r \right\}.$$

Note that $\exp(X) \equiv \exp(X; [\infty, N])$ and $\exp(X; (0)) \equiv \exp(X; (0, N))$ do not depend on $N$ and any polynomials are in $\exp(X; (0))$. Clearly,

$$\exp(X; (0)) \subset \exp(X; [r, N]) \subset \exp(X; (r, N)) \subset \exp(X).$$

For further details of these spaces, see [4].

For a space $X, X'$ means the dual space of $X$. For example, $O'(\tilde{B}_N(r))$ denotes the dual space of $O(\tilde{B}_N(r))$. $\langle , \rangle$ will denote the canonical inner product of $X'$ and $X$. When $T_z \in X'$ and $f(z) = \exp(z \cdot \zeta) \in X$, the Fourier-Borel transform of $T$ is defined by $\langle T_z, \exp(z \cdot \zeta) \rangle$. We call the mapping

$$T \mapsto \langle T_z, \exp(z \cdot \zeta) \rangle$$

the Fourier-Borel transformation. For $\zeta \in \tilde{B}_N(r)$ and $T \in \exp'(C^n; [r, N^*])$ we denote the Fourier-Borel transformation by $F$. A. Martineau [3] proved the following theorem:

**Theorem 1.** Let $N$ be a norm on $C^n$. The Fourier-Borel transformation $F$ establishes the following topological linear isomorphisms:

$$F : \exp'(C^n; [r, N^*]) \sim \rightarrow O(\tilde{B}_N(r)), \quad F : \exp'(C^n; (r, N^*)) \sim \rightarrow O(\tilde{B}_N[r]),$$

$$F : O'(\tilde{B}_N(r)) \sim \rightarrow \exp(C^n; [r, N^*]), \quad F : O'(\tilde{B}_N[r]) \sim \rightarrow \exp(C^n; (r, N^*)).$$

**2. Fourier-Borel Transformation on the Complex Sphere**

Let $\lambda \in C$. We call

$$\tilde{S}_\lambda = \{ z \in C^n; z^2 = z_1^2 + \cdots + z_n^2 = \lambda^2 \}, \quad \tilde{S}_\lambda, N(r) = \tilde{S}_\lambda \cap \tilde{B}_N(r), \quad \tilde{S}_\lambda, N[r] = \tilde{S}_\lambda \cap \tilde{B}_N[r]$$

the complex sphere. For $T \in \exp'(\tilde{S}_\lambda; [r, N^*])$ we denote the Fourier-Borel transformation by $F_\lambda$. Let $\Delta$ be the complex Laplacian: $\Delta_z = \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}$. For a set $D \subset C^n$ and $* = (r, N^*)$ or $[r, N^*]$, put

$$O_\lambda(D) = \{ f \in O(D); \Delta_z f(z) = \lambda^2 f(z) \}.$$

If $f \in O_\lambda(C^n; *) \equiv \exp(C^n; *) \cap O_\lambda(C^n)$.

For any $\zeta \in \tilde{S}_\lambda, N(r)$ and $T \in \exp'(C^n; [r, N^*])$ we can consider the spherical Fourier-Borel transformation $F^\lambda$ by

$$F^\lambda : T \mapsto F^\lambda T(\zeta) = \langle T_z, \exp(z \cdot \zeta) \rangle, \quad \zeta \in \tilde{S}_\lambda, N(r).$$

For the Lie norm given by $L(z) = (||z||^2 + \sqrt{||z||^4 - ||z^2||^2})^{1/2}$, where $||z||^2 = z \cdot \bar{z}$, R. Wada and the second named author proved the following theorem:
Theorem 2. Let $L$ be the Lie norm. The Fourier-Borel transformation $\mathcal{F}_\lambda$ establishes the following topological linear isomorphisms:

$$
\begin{align*}
\mathcal{F}_\lambda : \quad & \exp'(\tilde{S}_\lambda; [r, L^*]) \xrightarrow{\sim} \mathcal{O}_\lambda(\tilde{B}_L(r)), \quad \mathcal{F}_\lambda : \quad & \exp'(\tilde{S}_\lambda; (r, L^*)) \xrightarrow{\sim} \mathcal{O}_\lambda(\tilde{B}_L[r]), \\
\mathcal{F}_\lambda : \quad & \mathcal{O}'(\tilde{S}_{\lambda,L}(r)) \xrightarrow{\sim} \exp\lambda(\mathbb{C}^n; [r, L^*]), \quad \mathcal{F}_\lambda : \quad & \mathcal{O}'(\tilde{S}_{\lambda,L}[r]) \xrightarrow{\sim} \exp\lambda(\mathbb{C}^n; (r, L^*)).
\end{align*}
$$

Note that $L^*(z) = (||z||^2 + |z|^2)^{1/2}/\sqrt{2}$. In 1990's, we proved the following theorem:

Theorem 3. Let $L$ be the Lie norm. The spherical Fourier-Borel transformation $\mathcal{F}^\lambda$ establishes the following topological linear isomorphisms:

$$
\begin{align*}
\mathcal{F}^\lambda : \quad & \mathcal{O}'(\tilde{B}_L(r)) \xrightarrow{\sim} \exp(\tilde{S}_\lambda; [r, L^*]), \quad \mathcal{F}^\lambda : \quad & \mathcal{O}'(\tilde{B}_L[r]) \xrightarrow{\sim} \exp(\tilde{S}_\lambda; (r, L^*)), \\
\mathcal{F}^\lambda : \quad & \exp'^\lambda(\mathbb{C}^n; [r, L^*]) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_{\lambda,L}(r)), \quad \mathcal{F}^\lambda : \quad & \exp'^\lambda(\mathbb{C}^n; (r, L^*)) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_{\lambda,L}[r]).
\end{align*}
$$

2.1. On proofs of our theorems. First we proved Theorems 2 and 3 using the double series expansions and some properties of the Lie norm. Then in [1], we gave another proofs using Martineau’s theorem (Theorem 1), a well-known Oka’s theorem and theorems on homological algebra as follows: By the definition, we have the exact sequence

$$
0 \rightarrow \exp\lambda(\mathbb{C}^n; [r, L^*]) \xrightarrow{\iota} \exp(\mathbb{C}^n; [r, L^*]) \xrightarrow{\Delta - \lambda^2} \exp(\mathbb{C}^n; [r, L^*]).
$$

where $\iota$ is the canonical injection. By Oka’s theorem, we have the exact sequence

$$
0 \rightarrow \mathcal{O}(\tilde{B}_L(r)) \xrightarrow{(z^2 - \lambda^2)} \mathcal{O}(\tilde{B}_L[r]) \xrightarrow{\alpha_\lambda} \mathcal{O}(\tilde{S}_{\lambda,L}(r)) \rightarrow 0,
$$

where $(z^2 - \lambda^2)$ is the multiplication mapping and $\alpha_\lambda$ is the restriction mapping. Take the dual of (1) and apply Theorem 1. Then we have the following commutative and exact diagram:

$$
\begin{array}{cccc}
0 & \mathcal{O}'(\tilde{S}_{\lambda,L}(r)) & \mathcal{O}'(\tilde{B}_L(r)) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \exp\lambda(\mathbb{C}^n; [r, L^*]) & \exp(\mathbb{C}^n; [r, L^*]) & 0 \\
\end{array}
$$

where $\alpha_\lambda^*$ is the dual mapping of $\alpha_\lambda$. By Theorem 3.1 in [5] and Theorem 1, we also have the following commutative and exact diagram:

$$
\begin{array}{cccc}
0 & \mathcal{O}_\lambda(\tilde{B}_L(r)) & \mathcal{O}(\tilde{B}_L(r)) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \exp'(\tilde{S}_\lambda; [r, L^*]) & \exp'(\mathbb{C}^n; [r, L^*]) & 0 \\
\end{array}
$$

Applying theorems on homological algebra for the above commutative and exact diagrams, we have Theorem 2. Similarly we proved Theorem 3 which is the dual statement of Theorem 2. See [1] for the details.
2.2. First remark - A generalization of our theorems. Considering the above proofs, in the last two statements of Theorems 2 and 3, we do not have to restrict the norm to $L$ and $L^*$. For an analytic set $X \subset \mathbb{C}^n$, we put

\[
Exp(X; [r, N^*]) = \text{Exp}(\mathbb{C}^n; [r, N^*])|_X, \quad Exp(X; (r, N^*)) = \text{Exp}(\mathbb{C}^n; (r, N^*))|_X.
\]

When $* = (r, L^*)$ or $[r, L^*]$. $Exp(\tilde{S}_\lambda; *) = Exp(\tilde{S}_\lambda; *$) as a space. Here we introduce the topology of $Exp(X; *)$ for $* = (r, N^*)$ or $[r, N^*]$ by the quotient topology of the restriction mapping. Then we can generalize Theorems 2 and 3 as follows:

**Theorem 4.** Let $N$ be a norm on $\mathbb{C}^n$. The Fourier-Borel transformation $\mathcal{F}_\lambda$ establishes the following topological linear isomorphisms:

\[
\begin{align*}
\mathcal{F}_\lambda : \quad & \text{Exp}'(\tilde{S}_\lambda; [r, N^*]) \xrightarrow{\sim} \mathcal{O}_\lambda(\tilde{B}_N(r)), \\
\mathcal{F}_\lambda : \quad & \text{Exp}'(\tilde{S}_\lambda; (r, N^*)) \xrightarrow{\sim} \mathcal{O}_\lambda(\tilde{B}_N[r]), \\
\mathcal{F}_\lambda : \quad & \mathcal{O}'(\tilde{S}_\lambda_N[r]) \xrightarrow{\sim} \text{Exp}_\lambda(\mathbb{C}^n; [r, N^*]), \\
\mathcal{F}_\lambda : \quad & \mathcal{O}'(\tilde{S}_\lambda_N[r]) \xrightarrow{\sim} \text{Exp}_\lambda(\mathbb{C}^n; [r, N^*]).
\end{align*}
\]

**Theorem 5.** Let $N$ be a norm on $\mathbb{C}^n$. The spherical Fourier-Borel transformation $\mathcal{F}_\lambda$ establishes the following topological linear isomorphisms:

\[
\begin{align*}
\mathcal{F}_\lambda : \quad & \mathcal{O}'_{\lambda}(\tilde{B}_N(r)) \xrightarrow{\sim} \text{Exp}(\tilde{S}_\lambda; [r, N^*]), \\
\mathcal{F}_\lambda : \quad & \mathcal{O}'_{\lambda}(\tilde{B}_N[r]) \xrightarrow{\sim} \text{Exp}(\tilde{S}_\lambda; (r, N^*)). \\
\mathcal{F}_\lambda : \quad & \text{Exp}'_{\lambda}(\mathbb{C}^n; [r, N^*]) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_\lambda_N(r)), \\
\mathcal{F}_\lambda : \quad & \text{Exp}'_{\lambda}(\mathbb{C}^n; (r, N^*)) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_\lambda_N[r]).
\end{align*}
\]

3. FOURIER-BOREL TRANSFORMATION ON SOME VARIETIES

Recently, in [2], A. Kowata and M. Moriwaki considered the Fourier-Borel transformation on a variety $V_p = \{ z \in \mathbb{C}^n; p(z) = 0 \}$, where $p$ is a polynomial. Put

\[
\mathcal{O}_p(\mathbb{C}^n) = \{ f \in \mathcal{O}(\mathbb{C}^n); p(\partial z)f(z) = 0 \}.
\]

When $p(z) = z^2 - \lambda^2$, $p(\partial z)f(z) = (\Delta - \lambda^2)f(z)$ and $\mathcal{O}_p(\mathbb{C}^n) = \mathcal{O}(\mathbb{C}^n)$.

If a polynomial is given as a product of irreducible polynomials with no multiplicity, they call it a reduced polynomial. We denote by $\mathcal{F}_p$ the Fourier-Borel transformation for $T \in \text{Exp}'(V_p)$. They proved the following theorem:

**Theorem 6.** The Fourier-Borel transformation $\mathcal{F}_p : \text{Exp}'(V_p) \to \mathcal{O}_p(\mathbb{C}^n)$ is a topological linear isomorphism if and only if $p$ is a reduced polynomial.

3.1. Second remark - Our main theorems. Put

\[
V_{p,N}(r) = V_p \cap \tilde{B}_N(r), \quad V_{p,N}[r] = V_p \cap \tilde{B}_N[r], \quad \text{Exp}_p(\mathbb{C}^n; *) = \text{Exp}(\mathbb{C}^n; *) \cap \mathcal{O}_p(\mathbb{C}^n)
\]

for $* = (r, N^*)$ or $[r, N^*]$. If $\zeta \in V_{p,N}(r)$, then for $f(z) = \exp(z \cdot \zeta)$ we have $p(\partial z)f(z) = 0$. Therefore, for $\zeta \in V_{p,N}(r)$ and $T \in \text{Exp}_p(\mathbb{C}^n; [r, N^*])$ we can consider the $p$-Fourier-Borel transformation $\mathcal{F}_p$ by

\[
\mathcal{F}_p : T \mapsto \mathcal{F}_pT(\zeta) = (T_z, \exp(z \cdot \zeta)), \quad \zeta \in V_{p,N}(r).
\]

Then Theorem 6 will be generalized as follows:

**Theorem 7.** Let $N$ be a norm on $\mathbb{C}^n$. The Fourier-Borel transformations

\[
\begin{align*}
\mathcal{F}_p : \quad & \text{Exp}'(V_p; [r, N^*]) \to \mathcal{O}_p(\tilde{B}_N(r)), \\
\mathcal{F}_p : \quad & \text{Exp}'(V_p; (r, N^*)) \to \mathcal{O}_p(\tilde{B}_N[r]), \\
\mathcal{F}_p : \quad & \mathcal{O}'(V_{p,N}(r)) \to \text{Exp}_p(\mathbb{C}^n; [r, N^*]), \\
\mathcal{F}_p : \quad & \mathcal{O}'(V_{p,N}[r]) \to \text{Exp}_p(\mathbb{C}^n; (r, N^*)).
\end{align*}
\]

are topological linear isomorphisms if and only if $p$ is a reduced polynomial.
Theorem 8. Let $N$ be a norm on $\mathbb{C}^n$. The $p$-Fourier-Borel transformations

\[
\mathcal{F}^p : \mathcal{O}_p^r(\hat{B}_N(r)) \longrightarrow \text{Exp}(V_p; [r, N^*]), \quad \mathcal{F}^p : \mathcal{O}_p^r(\hat{B}_N[r]) \longrightarrow \text{Exp}(V_p; (r, N^*)).
\]

\[
\mathcal{F}^p : \text{Exp}_p^r(\mathbb{C}^n; [r, N^*]) \longrightarrow \mathcal{O}(V_p, N(r)), \quad \mathcal{F}^p : \text{Exp}_p^r(\mathbb{C}^n; (r, N^*)) \longrightarrow \mathcal{O}(V_p, N[r]),
\]

are topological linear isomorphisms if and only if $p$ is a reduced polynomial.

3.2. Outline of our proof. A proof of "$\Longleftrightarrow$" in Theorems 7 and 8 will be given in the same way as in §2.1. We can prove "$\Longrightarrow$" with the idea of the proof in "$\Longleftarrow$". For example, assume $\mathcal{F}_p : \text{Exp}^r(V_p; [r, N^*]) \sim \mathcal{O}_p(\hat{B}_N(r))$. Then since we know that the sequences $0 \rightarrow \mathcal{O}_p(\hat{B}_N(r)) \overset{\iota}{\longrightarrow} \mathcal{O}(\hat{B}_N(r)) \overset{p(\partial z)}{\longrightarrow} \mathcal{O}(\hat{B}_N(r))$ and $0 \rightarrow \text{Exp}(\mathbb{C}^n; [r, N^*]) \overset{\psi(z)}{\longrightarrow} \text{Exp}(\mathbb{C}^n; [r, N^*])$ are exact, we have the following commutative and exact diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}_p(\hat{B}_N(r)) & \xrightarrow{\iota} \mathcal{O}(\hat{B}_N(r)) & \xrightarrow{p(\partial z)} \mathcal{O}(\hat{B}_N(r)) & \rightarrow 0 \\
0 & \rightarrow & \text{Exp}^r(V_p; [r, N^*]) & \xrightarrow{\alpha_p^*} \text{Exp}^r(\mathbb{C}^n; [r, N^*]) & \xrightarrow{\psi(z)} \text{Exp}^r(\mathbb{C}^n; [r, N^*]) & \rightarrow 0
\end{array}
\]

where $\alpha_p^*$ is the dual mapping of the restriction mapping $\alpha_p$. Take the dual of the above diagram. The exactness of the sequence of the second row implies Ker $\alpha_p = \text{Im} p$; that is, $\{ F \in \text{Exp}(\mathbb{C}^n; [r, N^*]) : F|_{V_p} = 0 \} = \{ p(z)f(z) : f \in \text{Exp}(\mathbb{C}^n; [r, N^*]) \}$. If $p$ is not reduced, we can write $p = p_1^2 p_2$ ($p_1 \not= p_2$), where $p_1$ is irreducible and $p_2$ is reduced. Then $p_1 p_2 \in \{ F \in \text{Exp}(\mathbb{C}^n; [r, N^*]) : F|_{V_p} = 0 \}$ but $p_1 p_2 \not\in \{ p(z)f(z) : f \in \text{Exp}(\mathbb{C}^n; [r, N^*]) \}$ because $1/p_1 \not\in \text{Exp}(\mathbb{C}^n; [r, N^*])$. Thus, if $p$ is not reduced, then Ker $\alpha_p \not= \text{Im} p$.

We will omit the details of our proof, which might be given elsewhere.

REFERENCES


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