**ZONAL HERMITIAN MONOGENIC FUNCTIONS**

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**Abstract.** This paper deals with zonal spherical functions within Hermitian Clifford analysis, a function theory which is centred around two complex Dirac operators invariant with respect to the action of the unitary Lie algebra.

1. Introduction

Clifford analysis is nowadays regarded as a tool for studying differential operators, invariant with respect to the action of a Lie algebra $\mathfrak{g}$ and acting between functions taking values in irreducible $\mathfrak{g}$-modules, from the function theoretical point of view. In order to properly define the principal objects studied in Clifford analysis, one needs to combine methods from representation theory and differential geometry. However, whereas this leads to abstractly defined operators acting between sections of associated principal fibre bundles on general manifolds, Clifford analysis techniques are applied to local explicit realizations in which the operators arise as endomorphisms on the Clifford algebra.

Until recently, in Clifford analysis one mostly studied invariant operators with respect to the Lie algebra $\mathfrak{so}(m)$: e.g. the Dirac operator, acting on spinor-valued functions, the Rarita-Schwinger operator and its higher-spin generalizations. For the Dirac operator, the most widely studied object in Clifford analysis, we refer to the standard references [1, 6, 8], whereas for the other operators we refer to e.g. [4, 5, 14]. This has lead to a well-developed theory, to which we refer as the classical orthogonal setting. Recently, Clifford analysis techniques have been applied to differential operators invariant with respect to the complex Lie algebra $\mathfrak{su}(m)$, giving rise to what is called complex Hermitian Clifford analysis, see [2, 3, 7, 9, 10]. This paper has to be situated within the complex setting, centred around two complex Dirac operators appearing naturally in the framework of Kähler geometry for complex manifolds. Our aim is to define the so-called zonal spherical functions within the Hermitian setting, playing a crucial role in the decomposition of spaces of complex polynomials into $\mathfrak{su}(m)$-irreducible summands. We define these zonal Hermitian monogenic functions using two different techniques: one of them is closer to the spirit of the orthogonal setting, which allows to draw analogies with the classical case, whereas the other is essentially based on a representation theoretical observation, see e.g. [13]. This is crucial because the term zonal has a peculiar meaning within the representation theoretical language, which we would like to enlighten in this paper.

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2. HERMITIAN CLIFFORD ANALYSIS

In this section we briefly introduce the main concepts, for a more elaborate introduction to Hermitian Clifford analysis, we refer e.g. to [2, 7]. Let $\mathbb{C}^{2m} = W^+ \oplus W^-$ be the isotropic Witt decomposition of the complex orthogonal space $\mathbb{C}^{2m}$ endowed with the standard Euclidean bilinear form $B_C(<.,.>)$, where $W^+ = \text{span}_\mathbb{C}\{f_j : 1 \leq j \leq m\}$ and $W^- = \text{span}_\mathbb{C}\{f_j^\dagger : 1 \leq j \leq m\}$. These subspaces may be defined as the eigenspaces of a complex structure $J \in SO_C(2m)$, with eigenvalues $\pm i$. The Clifford algebra $\mathbb{C}_2m$ is generated by the Witt basis vectors $f_j$ and $f_j^\dagger$ by means of the following relations: $f_j f_k^\dagger + f_k f_j^\dagger = \delta_{jk}$ and $f_j^2 = (f_j^\dagger)^2 = 0$. The complex Dirac operators are then defined as the vector-valued differential operators $\partial_z = \sum_j f_j \partial_{z_j}$ and $\partial_{z_j}^\dagger = \sum_j f_j \partial_{z_j}$, where the Cauchy-Riemann operators are associated to the complex coordinates defined by the complex structure $J$, and functions belonging to the kernel of both operators are called $h$-monogenic functions. In view of the fact that these operators are invariant with respect to a subgroup of the spin group $\text{Spin}(2m)$ which is isomorphic to a double cover for the unitary Lie group $U(m)$, see e.g. [2], the study of $h$-monogenic functions is related to $\mathfrak{sl}(m)$-invariant modules, see [7]. A main role is played by homogeneous polynomials, i.e. eigenfunctions for the complex Euler operators $E_z = \sum_j z_j \partial_{z_j}$ and $E_z^\dagger = \sum_j z_j \partial_{z_j}^\dagger$, measuring the degree of homogeneity in the complex vector variables.

3. ZONAL FUNCTIONS IN CLIFFORD ANALYSIS

The Fischer decomposition in harmonic analysis can be refined to the case of spinor-valued homogeneous polynomials as follows: given the polynomial $R_k(X)$ of degree $k$ in $(X_1, \ldots, X_{2m})$, it is well-known that $R_k(X) = \sum_j X^{k-j} P_j(X)$, where $P_j(X) \in \ker(\partial_X)$ denotes a $\partial_X$-monogenic $j$-homogeneous polynomial, see e.g. [6], with $\partial_X$ the classical Dirac operator. On the other hand, we also have that

$$ R_k(X) = \frac{1}{k!} <X, \partial_Y>^k R_k(Y) = \frac{1}{k!} \left( <X, Y>^k, R_k(Y) \right). \tag{1} $$

The reproducing kernel $<X, Y>^k$ is zonal, which means that it depends on the inner product of $X$ and $Y$ only. The Fischer decomposition in both variables $X$ and $Y$ can then be written as $<X, Y>^k = \sum_j X^j Z_{k,j}(X, Y) Y^j$, where each $Z_{k,j}(X, Y)$ stands for a $(k-j)$-homogeneous zonal polynomial for which $\partial_X Z_{k,j}(X, Y) = 0 = Z_{k,j}(X, Y) \partial_Y$. This illustrates the importance of having homogeneous double-sided zonal monogenics. Using a recursive argument, it is easily seen that $Z_{k,j} = C_{k,j} Z_{k-j,0}$ for a specific constant. To construct $Z_{k,0}(X, Y)$, it then suffices to determine a homogeneous zonal scalar-valued function $S_k(X, Y) = |X|^k |Y|^k S_k(t)$ which is harmonic in $X$ and $Y$, where $t$ stands for the normalized inner product: $<X, Y> = t|X||Y|$. These conditions on $S_k(X, Y)$ give rise to Gegenbauer’s differential equation for $S_k(t)$, which means that the zonal polynomials $Z_{k,0}(x, y)$ can be determined as follows:

$$ Z_{k,0}(X, Y) = c_k \partial_X \left( |X|^{k+1} C_{k+1,0}^{m-1}(t) |Y|^{k+1} \right) \partial_Y, $$

where $c_k$ denotes a constant depending on $k$, see [11]. An obvious way to generalize the role of the functions $Z_k(X, Y)$ to the Hermitian setting is the following: by analogy with the orthogonal case, in which zonal double-sided monogenic functions are obtained from
a zonal biharmonic function which is symmetric in all variables, we will here start from a zonal complex harmonic function which is symmetric in the four variables \((z, z^\dagger; u, u^\dagger)\). However, as opposed to the orthogonal case in which the notion zonal has a distinct geometrical meaning, it is not a priori clear how to define the notion of being zonal in the Hermitian setting. Hence the following definition:

**Definition 1.** A function \(f(z, z^\dagger; u, u^\dagger) : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \rightarrow \mathbb{C}\) is Hermitian zonal if it depends on the Hermitian inner products \(\{z, z^\dagger\}, \{u, u^\dagger\}, \{z, u^\dagger\}\) and \(\{u, z^\dagger\}\).

In the first place, we are thus looking for scalar-valued functions \(f\) such that

\[
\Delta_z \left(f(\{z, z^\dagger\}, \{u, u^\dagger\}, \{z, u^\dagger\}, \{u, z^\dagger\})\right) = 0 .
\]

If we moreover find solutions which are symmetric under \((z, z^\dagger) \leftrightarrow (u, u^\dagger)\), it is clear that one can use these functions to construct double-sided monogenics by means of the action of the Hermitian Dirac operators from both sides. In view of the fact that arbitrary \((k, l)\)-homogeneous polynomials in \((z, z^\dagger)\) can be obtained by means of the reproducing kernel formula \(k!! R_{k,l}(u, u^\dagger) = \{u, \partial_z\}^k \{u^\dagger, \partial_z^\dagger\}^l R_{k,l}(z, z^\dagger)\), which is the Hermitian refinement of the orthogonal reproducing formula, we propose the following normalized Hermitian zonal variable :

\[
s = \frac{\{z, u^\dagger\} \{z^\dagger, u\}}{\{z, z^\dagger\} \{u, u^\dagger\}} .
\]

Consider then an arbitrary \((k, l) \in \mathbb{N}^2\), and suppose that \(k \geq l\). In view of our strategy, there is a canonically determined zonal scalar \((k, l)\)-homogeneous function which can lead to double-sided \(h\)-monogenics of degree \((k - 1, l - 1)\) :

\[
S_{k,l}(z, z^\dagger; u, u^\dagger) = \{z, u^\dagger\}^{k-l} \{z^\dagger, u\}^l S_{k,l}(s) \{u, u^\dagger\}^l .
\]

Letting the complex Laplacian \(\Delta_z\) act on this zonal scalar function, we are lead to the following differential equation for the function \(S_l(s)\) :

\[
s(1-s)S''_l + ((1+k-l)-(m+k-l)s)S'_l + l(k+m-1)S_{k,l} = 0 .
\]

This is the hypergeometric differential equation, for which two independent solutions exist. Only one of those leads to a polynomial, given explicitly in terms of the Jacobi polynomials \(P_n^{(\alpha, \beta)}(t)\) by

\[
S_{k,l}(s) = (-1)^l \frac{l!(k-l)!}{k!} P_l^{(m-2,k-l)}(2s-1) .
\]

(2)

Note that the Gegenbauer polynomials, playing a crucial role in the orthogonal setting, are particular examples of the Jacobi polynomials for specific values of the parameters \((\alpha, \beta; n)\). This illustrates nicely that the Hermitian setting refines the classical setting. We have thus found the following Hermitian zonal two-sided \(h\)-monogenic function:

\[
Z_{k,l}(z, z^\dagger; u, u^\dagger) = \partial_{\bar{z}}^l \partial_z (S_{k+1,l+1}(z, z^\dagger; u, u^\dagger)) \partial_{\bar{u}}^l \partial_u .
\]

Note that the order of the differential operators is not a priori described, but in view of the fact that the function between brackets is a complex harmonic function, it is immediately clear that changing the order will only give a minus sign. In case \(k < l\), it suffices to switch the roles of \((z, u^\dagger)\) and \((z^\dagger, u)\).
To prove that the functions \( S_{k,l}(z, z^l; u, u^l) \) are good Hermitian analogues of the orthogonal functions \( Z_k(X, Y) \), we can use the following argument. The space \( \mathcal{H}_k^{2m} \) of \( k \)-homogeneous harmonic polynomials in \( 2m \) variables \( (X_1, \ldots, X_{2m}) \) decomposes as an \( \mathfrak{su}(m) \)-module into a direct sum \( \mathcal{H}_k^{2m} = \oplus_j \mathcal{H}_j^{k-j} \), where each space \( \mathcal{H}_j^{k-j} \) contains \((a, b)\)-homogeneous harmonic polynomials in \((z, z^l)\). As shown in [7], these spaces are \( \mathfrak{su}(m) \)-irreducible and have highest weight \((b + a, a, \ldots, a) \in \mathbb{Z}^{m-1}\). If we now apply this to the function \( |X|^k |Y|^k C_{k}^{-1}(t) \in \mathcal{H}_k^{2m} \), we have that

\[
|X|^k |Y|^k C_{k}^{-1}(t) = \sum_{j=0}^{k} C_{j,k-j}^{-1}(z, z^l; u, u^l) \in \bigoplus_{j=0}^{k} \mathcal{H}_j^{k-j}.
\]  

(3)

Using straightforward calculations, exploiting some identities for special functions, it is easily seen that \( C_{j,k-j}^{-1}(z, z^l; u, u^l) \) is given by one of the Hermitian zonal functions determined above, up to a constant. In case \( k = 2l \) for example, we get:

\[
C_{l+j,l-j}^{-1}(z, z^l; u, u^l) = \left( \{z, z^l\} \{u, u^l\} \right)^{l-j} \{(z, z^l)\}^{2j} P_{l-j}(s)
\]

\[
C_{l-j,l+j}^{-1}(z, z^l; u, u^l) = \left( \{z, z^l\} \{u, u^l\} \right)^{l-j} \{(z^l, u^l)\}^{2j} P_{l-j}(s)
\]

with \( P_{l-j}(s) \sim S_{l+j,l-j}(s) \) (for all \( 0 \leq j \leq l \)), as desired.

4. ZONAL FUNCTIONS AND GROUP REPRESENTATIONS

The aim of this section is to compare our results from the previous section with the results obtained in [13]. The main reason for doing so is to obtain a better meaning of the notion zonal from the representation theoretical point of view. This way, we will be able to prove that our somewhat intuitive guess for the Hermitian zonal argument \( s \) does indeed lead to the zonal functions as obtained using the language of [13]. Moreover, it clarifies the geometrical meaning of this argument \( s \). Let \( T : G \to \text{Aut}(V) \) be an irreducible representation of a (possible non-compact) group \( G \) on a Hilbert space \( V \), and let \( H \subseteq G \) be a subgroup. A vector \( w \in V \) is invariant w.r.t. \( H \) if \( T(h)[w] = w \) for all \( h \in H \). If there exists such a \( w \in V \) and all operators \( T(h) \) are unitary, \( T \) is said to be a representation of class 1 w.r.t. \( H \). For normalized \( w \), one can consider the space of functions \( \mathcal{M}_{T,w} := \{ F_z : g \in G \mapsto (T(g)[z], w) \mid z \in V \} \). Such functions are by definition spherical functions. Since the operators \( T(h) \) are unitary, we clearly have \( f(hg) = f(g) \), for all \( h \in H \). This means that spherical functions are constant on cosets \( Hg \), whence they can be regarded as functions on a right homogeneous space \( X = H \setminus G \). Choosing a basis \( \{e_i\} \) in \( V \), such that \( e_1 = w \), we have that the representation \( T \) corresponds to a matrix. By definition, the matrix element \( t_{11}(g) = (T(g)e_1, e_1) \) is called the zonal spherical function of \( T \) w.r.t. \( H \) and the matrix elements \( t_{1i}(g) = (T(g)e_i, e_1) \) are called associated spherical functions. Note that the latter functions span the space \( \mathcal{M}_{T,w} \), and that the zonal spherical functions are constant on two-sided cosets \( HgH \).

Let us then consider the group \( U(m) \), and choose \( V = L^2(S^{m-1}) = \bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \mathcal{H}_{k,l}^{m} \), with \( S^{m-1} \) the homogeneous space \( U(m)/U(m-1) \) containing \( z(z_1, \ldots, z_m) \in \mathbb{C}^m \) for which \( \{z, z^l\} = 1 \) and where \( \mathcal{H}_{k,l}^{m} \) contains restrictions to \( S^{m-1} \) of complex harmonic polynomials which are \((k, l)\)-homogeneous in \((z, z^l)\). The standard coordinates on \( S^{m-1} \)
are chosen in a way which resembles the polar coordinates on the sphere $S^{m-1}$:

$$
\ominus (\omega_1, \cdots, \omega_m) \in S^{m-1}_C \Leftrightarrow \begin{cases}
\omega_1 = e^{i\varphi_1} \sin \theta_{m-1} \cdots \sin \theta_2 \sin \theta_1 \\
\omega_2 = e^{i\varphi_2} \sin \theta_{m-1} \cdots \sin \theta_2 \cos \theta_1 \\
\vdots \\
\omega_{m-1} = e^{i\varphi_{m-1}} \sin \theta_{m-1} \cos \theta_{m-2} \\
\omega_m = e^{i\varphi_m} \cos \theta_{m-1}
\end{cases}
$$

The Hilbert spaces $\tilde{H}^m_{k,l}$ carry irreducible representations $T^m_{k,l}$ of $U(m)$, defined by

$$
T^m_{k,l}(g) \left[ f(z, z^\dagger) \right]_{S^{m-1}_C} = f(g^{-1}z, g^{-1}z^\dagger) \mid_{S^{m-1}_C}.
$$

Applying branching rules for $U(m)$ to the chain $U(m) \supset U(m-1) \supset \cdots \supset U(1)$ and using the fact that irreducible $U(1)$-representations are one-dimensional, the decomposition above can be refined as follows: $L^2(S^{m-1}_C) = \bigoplus_{k,l=0}^{\infty} \tilde{H}^m_{k,l;O}$, where $M$ is a vector consisting of integers satisfying certain conditions following from the abstract branching rules. One may then fix unit vectors $\Xi^m_{k,l;O}$ in these one-dimensional subspaces, and the set $\{\Xi^m_{k,l;O}\}$ forms an orthonormal basis for $\tilde{H}^m_{k,l}$ for fixed $k$ and $l$. Denoting matrix elements by means of $t^m_{M,N}(g)$, the spherical functions are found by restricting to the case $N = O$. The zonal spherical functions of the representation $T^m_{k,l}$ for $U(m)$ w.r.t. $U(m-1)$ are then found as the functions $t^m_{M,N}(g) = \langle t^m_{k,l}(g) | \Xi^m_{k,l;O} \rangle$, which are constant on two-sided cosets w.r.t. $U(m-1)$. Invoking the $KAK$-decomposition, which in case of the group $U(m)$ reads as $U(m) = KU(1)AK$ with $K = U(m-1)$ and where $A$ represents the one-parameter subgroup generated by rotations of the form $g_{m-1}(\theta)$, leads to

$$
t^m_{M,N}(g) = t^m_{M,N}(k_1 g_{m-1}(\theta) k_2) := e^{i(k-1)\phi} \varphi^{m,k,l}(\theta)
$$

It is then proved in [13] that

$$
\varphi^{m,k,l}(\theta) = \frac{l!(m-2)!}{(l+m-2)!} (\cos \theta)^{k-l} P_l^{(m-2,k-l)}(\cos 2\theta).
$$

where $P_n^{(a,b)}$ represents the Jacobi polynomial. Comparing this expression with what was found in section 3. see equation (2), it is clear that our Hermitian zonal variable $s$ can be expressed in terms of the angles used in [13]. Up to a constant we have the following correspondence:

$$
\{z, u^\dagger\}^k l P_l^{(m-2,k-l)}(2s-1) \iff (e^{i\phi} \cos \theta)^{k-l} P_l^{(m-2,k-l)}(\cos 2\theta).
$$

From this it immediately follows that $\{z, u^\dagger\} = e^{i\phi} \cos \theta$. This means that the Hermitian inner product uniquely determines the angles appearing in the $KAK$-decomposition for $U(m)$ in the language of [13]. Moreover, we also have that

$$
\{z, u^\dagger\}^\dagger = \{z^\dagger, u\} \iff \{z^\dagger, u\} = e^{-i\phi} \cos \theta.
$$

Looking at the identification with $s$, we indeed clearly have that

$$
2s - 1 = \cos 2\theta \iff s = \cos^2 \theta = \frac{\{z, u^\dagger\} \{z^\dagger, u\}}{\{z, z^\dagger\} \{u, u^\dagger\}}
$$

where we have used that $\{z, z^\dagger\} = \{u, u^\dagger\} = 1$ on $S^{m-1}_C$. 


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REFERENCES


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