SPECTRAL INVARIANT FRÉCHET ALGEBRAS IN THE THEORY OF TOEPLITZ OPERATORS

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ABSTRACT. We recall the notion of $\Psi^*$-algebras introduced in [7] resp. of algebras closed under holomorphic functional calculus. Some basic properties and examples are provided. Spectral invariant algebras can be defined by commutator methods. We demonstrate such a procedure by constructing localized versions $\mathcal{A}$ of the Hoermander classes $\Psi^0_{\rho,\delta}$ of 0-order pseudo-differential operators. By specializing to the exotic case $\rho = \delta = \frac{1}{2}$ and using a result in [11] we show that Toeplitz operators on the Heisenberg group induced from the Hardy space over the unit ball in $\mathbb{C}^{n+1}$ are contained in these $\Psi^*$-algebras $\mathcal{A}$.

1. INTRODUCTION

As one fundamental step to provide a suitable framework for analyzing a ring $R$ of operators, the construction of an embedding into a closed operator algebra $\mathcal{A}$ with good topological and algebraic properties is of importance. Moreover, $\mathcal{A}$ should be as small as possible and reflect some structure of the elements in $R$. For dealing with operators which have local properties (e.g. micro or pseudo locality) the concept of $C^*$-algebras is not as useful since - roughly speaking - such properties are not preserved under taking norm closures. The appropriate replacement is given by the notion of a $\Psi^*$- resp. $\Psi_0$-algebra introduced through B. Gramsch in [7]. The first type is spectrally invariant in an enveloping space and carries a refined Fréchet topology. During the last two decades it has attracted attention that many consequences arise for algebras of this kind. In the present note we explain a way how to construct enveloping $\Psi^*$-algebras for a given ring of Toeplitz operators. In our analysis we use a result in [11] due to A. Nagel and E.M. Stein which states that the Szegoe-projection for the Hardy space over the real $2n + 1$-dimensional unit sphere is a pseudo-differential operator of exotic type $(\frac{1}{2}, \frac{1}{2})$.

2. ALGEBRAS CLOSED UNDER HOLOMORPHIC FUNCTIONAL CALCULUS

Let $(\mathcal{B}, \| \cdot \|_B)$ be a Banach algebra with unit $e$ and $\mathcal{A} \subset \mathcal{B}$ a subalgebra which not necessarily has to be unital. In case $\mathcal{A}$ is with unit we write $\mathcal{A}^{-1}$ for the group of invertible elements in $\mathcal{A}$. Given a compact set $K \subset \mathbb{C}$ we mean by $\mathcal{O}(K)$ the algebra of germs of holomorphic functions defined in a neighbourhood of $K$.

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Definition 1. The algebra \( \mathcal{A} \) is said to be closed under holomorphic functional calculus (CuHFC) in \( \mathcal{B} \) if for all \( a := \lambda e + x \in Ce + \mathcal{A} \) and \( f \in \mathcal{O}(\sigma_B(a)) \) the element \( f(a) \) which is defined via holomorphic functional calculus in \( \mathcal{B} \) is contained in \( \mathcal{A} \). Here \( \sigma_B(a) \) denotes the spectrum of \( a \) in \( \mathcal{B} \). Moreover, \( \mathcal{A} \) is called spectrally invariant in \( \mathcal{B} \) if
\[
( Ce + \mathcal{A} ) \cap \mathcal{B}^{-1} = ( Ce + \mathcal{A} )^{-1}.
\]

For details we refer to [7], [8], [10] and we only mention that the notion of CuHFC is stable under arbitrary intersections. Moreover, \( \mathcal{A} \) CuHFC in \( \mathcal{B} \) implies spectral invariance of \( \mathcal{A} \) in \( \mathcal{B} \). In the sequel we are concerned with algebras of pseudo-differential type operators. Additional assumptions on the topological structure of the embedding \( \mathcal{A} \hookrightarrow \mathcal{B} \) are crucial to model \( C^\infty \)-phenomena in an abstract sense.

Definition 2. (cf. [7], [8], [10]) Let \( \mathcal{B} \) be a unital \( C^* \)-algebra and \( A \subset B \) a symmetric subalgebra (i.e. \( a \in A \implies a^* \in A \)). Then \( A \) is called a \( \Psi^* \)-algebra in \( B \) if:

- \( A \) is spectrally invariant in \( B \) with \( e \in A \)
- \( A \) carries a Fréchet topology \( \tau_A \) and the embedding \( (A, \tau_A) \hookrightarrow \mathcal{B} \) is continuous.

\( A \) is said to be submultiplicative if \( \tau_A \) is generated by a sequence of submultiplicative semi-norms \( \{ q_j \}_{j \in \mathbb{N}} \), i.e. \( q_j(xy) \leq q_j(x) \cdot q_j(y) \) and \( q_j(e) = 1 \) for \( j \in \mathbb{N} \).

If \( A \) is a \( \Psi^* \)-algebra, then \( A^{-1} \) is open and the inversion \( A^{-1} \ni x \mapsto x^{-1} \in A \) is continuous. There is a local version of spectral invariance: Let \( \varphi : A \to B \) be a morphism of algebras where \( \varphi(e) = e \) in the case \( A \) is unital.

Definition 3. (cf. [7], [8], [10]) \( A \) is said to be locally spectral invariant with respect to \( \varphi \), if there is \( \varepsilon > 0 \) such that:
\[
\{ e + \varphi(x) \}^{-1} \in Ce + \varphi(A)
\]
for all \( x \in A \) with \( \| \varphi(x) \|_B \leq \varepsilon \). \( A \) is called locally spectral invariant if \( \varphi = id \).

Let \( \varphi : A \to B \) be one-to-one and assume that \( A \) is unital. In case where

- \( \varphi(A) \) is dense in \( B \)
- \( B \) is a \( C^* \)-algebra and \( \varphi(A) \) is symmetric in \( B \)

it is well-known (cf. [10]) that \( A \) is locally spectral invariant with respect to \( \varphi \) if and only if \( \varphi^{-1}(B^{-1}) = A^{-1} \) (which in the first place seems to be a stronger condition).

Note that the open mapping theorem can be interpreted as a theorem on spectral invariance. Here we recall other model cases. Given Hilbert spaces \( H, V \) we denote by \( \mathcal{L}(H, V) \) the bounded operators from \( H \) to \( V \). As usual we write \( \mathcal{L}(H) := \mathcal{L}(H, H) \).

(a): Let \( M \) be a compact manifold and \( \mathcal{P} \) an elliptic pseudo-differential operator of order \( m \geq 0 \) on \( M \) invertible as an (unbounded) operators on \( L^2(M) \) - the space of square integrable 1/2-densities on \( M \). Then \( \mathcal{P}^{-1} \) is a pseudo-differential operator again.

(b): Let \( 0 \leq \delta \leq \rho \leq 1 \) and \( \delta < 1 \). The Hoermander classes \( \Psi^0_{\rho, \delta} \) of 0-order pseudo-differential operators form a \( \Psi^* \)-algebra via the embedding \( \Psi^0_{\rho, \delta} \hookrightarrow \mathcal{L}(L^2(\mathbb{R}^n)) \), cf. [9].
(c): An element $a \in \mathcal{L}(H)$ is called relatively invertible in $\mathcal{L}(H)$ iff its range $R(a)$ is closed, i.e. there is a Moore-Penrose inverse $\hat{a} \in \mathcal{L}(H)$ s.t. $\hat{a}a = a$ and $\hat{a}a\hat{a} = \hat{a}$. In this case $\lambda = 0$ is an isolated point of the spectrum $\sigma(a^*a)$ of $a^*a$. The orthogonal projection $p$ onto the kernel $N(a^*a) = N(a)$ is given via functional calculus by:

$$p := \frac{1}{2\pi i} \int_{\gamma} (z \cdot \text{id}_H - a^*a)^{-1} dz$$

where $\gamma$ is a circle around $0 \in \mathbb{C}$ not intersecting $\sigma(a^*a)$ and $\hat{a} = (p + a^*a)^{-1}a^*$. If $\mathcal{A}$ is symmetric and CuHFC, then $a \in \mathcal{A} \implies \hat{a} \in \mathcal{A}$. The relatively invertible elements in a $\Psi^*$-algebra form a so called locally rational Fréchet manifold.

3. Commutator methods

Typically, proving spectral invariance resp. the $\Psi^*$-property in explicit examples is not an easy problem. However, there are several procedures to construct $\Psi^*$-algebras in an a-priori way. Here we recall a commutator method which is a flexible tool and allows to define $\Psi^*$-algebras with prescribed properties. Let $S(\mathbb{R}^n)$ be the Schwartz space and $S' (\mathbb{R}^n)$ denotes the tempered distributions. Moreover, with $s \in \mathbb{R}$ we write $H^s$ for the $s$th-Sobolev space. We start with a structural result which gives a characterization of the Hoermander classes above by commutator conditions, cf. [5].

Let $M$ be a linear space and $L(M)$ the algebra of all linear operators on $M$. For $X, B \in L(M)$ the commutator

$$\text{ad}[X](B) := [X, B] = XB - BX$$

defines a derivation on $L(M)$. Generalizing (1) to a finite system $S_{m+1} := [X_1, \cdots, X_{m+1}]$ of operators $X_j \in L(M)$ we inductively define the iterated commutators:

$$\text{ad}[S_{m+1}](B) := \text{ad}[X_{m+1} \left( \text{ad}[S_m](B) \right)] .$$

Furthermore, we shortly write $\text{ad}^l[X](B) := \text{ad}[X, \cdots, X](B)$ ($l$ times). In particular, for $\alpha, \beta \in \mathbb{N}_0^n$ and with $D_{x_j} := -i\frac{\partial}{\partial x_j}$ and $M = S(\mathbb{R}^n)$:

$$\text{ad}^\alpha[-ix] \text{ad}^\beta[D_{x_j}] := \text{ad}^{\alpha_1}[\cdots -ix_1 \cdots] \cdots \text{ad}^{\alpha_n}[\cdots -ix_n] \text{ad}^{\beta_1}[D_{x_1}] \cdots \text{ad}^{\beta_n}[D_{x_n}].$$

The following characterization of the Hoermander classes has been used to prove their spectral invariance:

**Theorem 1.** (Beals, [5]) Let $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. $B \in S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ be a continuous operator. Then, $B \in \Psi_{p, \delta}^\rho$ if and only if the commutators

$$\text{ad}[-ix]^\alpha \text{ad}[D_{x_j}]^\beta(B) : H^{s-\rho|\alpha|+\delta|\beta|} \rightarrow H^s$$

are well-defined as continuous extensions from $S(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}_0^n$ and $s \in \mathbb{R}$.

We demonstrate the commutator method by constructing localized versions of the Hoermander classes which are spectrally invariant or even define $\Psi^*$-algebras in $\mathcal{L}(L^2(\mathbb{R}^n))$. We start recalling some definitions, all proofs can be found in [1].
Fix \( m \in \mathbb{R} \) and \( 0 \leq \delta \leq \rho \leq 1, \delta < 1 \). To \( a \) in the symbol class \( S_{\rho,\delta}^m \) one associates the pseudo-differential operator \( R_a \) on \( S(\mathbb{R}^n) \) in the sense of [9]. As usual we write
\[
\Psi_{\rho,\delta}^m := \left\{ R_a : a \in S_{\rho,\delta}^m \right\} \quad \text{and} \quad \Psi^{-\infty} := \bigcap_{m \in \mathbb{N}} \Psi_{\rho,\delta}^m
\]
where \( \Psi^{-\infty} \) is independent from \( \rho \) and \( \delta \). For \( s \in \mathbb{R} \) an operator \( R_a \in \Psi_{\rho,\delta}^m \) continuously extends to \( \tilde{R}_a : H^{s+m} \to H^s \). Moreover, for \( P_j \in \Psi_{\rho,\delta}^{m_j}, j = 1, 2 \) one has:
\[
P_1 P_2 \in \Psi_{\rho,\delta}^{m_1+m_2} \quad \text{and} \quad [P_1, P_2] \in \Psi_{\rho,\delta}^{m_1+m_2-(\rho-\delta)}.
\] (3)

Let \( D(\mathbb{R}^n) \) be the space of smooth complex valued functions on \( \mathbb{R}^n \) having compact support. With the Euclidean norm \( \| \cdot \| \) on \( \mathbb{R}^n \) we write \( \lambda(x, \xi) := \sqrt{1 + \| \xi \|^2} \) and we define the operator \( \Lambda^s := R_{\lambda^s} \in \Psi_{1,0}^s \). To \( \Phi \in D(\mathbb{R}^n) \) we assign:
\[
\Lambda_{\Phi} := \text{Id} + \Phi \cdot \Lambda^{\frac{1}{2}} \cdot \Phi \in \Psi_{1,0}^\frac{1}{2}
\] (4)
where in our notation \( \Phi \) is identified with the multiplication by \( \Phi \). It can be shown that (4) admits a self-adjoint extension (Friedrichs-extension) with domain \( D(\Lambda_{\Phi}) \) of definition. A localized Sobolev space is now defined by:
\[
H^s_{\Phi} := \text{completion of } D(\Lambda_{\Phi}^{2s}) \text{ w.r.t. } \| \Lambda_{\Phi}^{2s} \cdot \|_{L^2(\mathbb{R}^n)}.
\]

For \( k \in \mathbb{N}_0 \) the embeddings \( H^s_{\Phi} \hookrightarrow H^s_{\Phi} \) are well-defined and continuous. We set:
- \( H^s_{\Phi} := \bigcap_{s \in \mathbb{R}} H^s_{\Phi} \) (projective topology)
- \( H^{-\infty}_{\Phi} := \bigcup_{s \in \mathbb{R}} H^s_{\Phi} \) (inductive topology).

**Proposition 1.** Let \( s \in \mathbb{R} \) and \( \Phi \in D(\mathbb{R}^n) \). For \( R \in \Psi_{\rho,\delta}^0 \) the commutators w.r.t. the multiplication by \( \Phi \) resp. \( \Lambda^s \) are of class:
\[
[\Phi, R] \in \Psi_{\rho,\delta}^{-\rho} \quad \text{and} \quad [\Lambda^s, R] \in \Psi_{\rho,\delta}^{-s(1-\delta)}.
\] (5)

Note that (5) shows an order (=the upper index) improvement of commutators in special cases even if \( R \) belongs to an exotic class \( \Psi_{\rho,\rho}^0 \) (whereas the general formula (3) does not). Having our applications in mind we are concerned with the case \( \rho = \delta = \frac{1}{2} \).

**Corollary 1.** Let \( R \in \Psi_{\frac{1}{2},\frac{1}{2}}^0 \). Then \( \text{ad}^l[\Lambda_{\Phi}](R) \in \Psi_{\frac{1}{2},\frac{1}{2}}^0 \) holds for all orders \( l \in \mathbb{N}_0 \). In particular, \( \text{ad}^l[\Lambda_{\Phi}](R) \) has a continuous extensions to \( L^2(\mathbb{R}^n) \).

**Proof:** It is sufficient to prove \( [\Lambda_{\Phi}, R] \in \Psi_{\frac{1}{2},\frac{1}{2}}^0 \). By Proposition 1 and using the inclusion \( \Psi_{1,0}^s \subset \Psi_{\frac{1}{2},\frac{1}{2}}^s \), cf. [9]:
\[
[\Lambda_{\Phi}, R] = \Phi \Lambda_{\Phi}^\frac{1}{2} [\Phi, R] + \Phi \left[ \Lambda_{\Phi}^\frac{1}{2}, R \right] \Phi + [\Phi, R] \Lambda_{\Phi}^\frac{1}{2} \Phi.
\]
The second assertion follows by a result due to Calderon/Vaillancourt in [4]. □
Fix $\Phi \in \mathcal{D}(\mathbb{R}^n)$ with $\Phi \geq 0$ and consider the operator algebra:

$$\mathcal{A}_\Phi := \left\{ a \in \mathcal{L}(L^2(\mathbb{R}^n)) : a(H^\infty_\Phi) \subset H^\infty_\Phi, \quad \forall j \in \mathbb{N}_0, \exists \alpha_j > 0 \right\}$$

$$\| \text{ad}^j [\Lambda_\Phi] (a) f \|_{L^2(\mathbb{R}^n)} \leq \alpha_j \cdot \| f \|_{L^2(\mathbb{R}^n)}, \quad \forall f \in H^\infty_\Phi \right\}. \quad (6)$$

To state some properties of $\mathcal{A}_\Phi$ we define the operators of order shift $\rho > 0$:

$$\bigcap_{t \in \mathbb{R}} \mathcal{L}(H^\rho_\Phi, H^{-\rho}_\Phi) := \left\{ a \in \mathcal{L}(H^\infty_\Phi) : \text{extends to } a_t \in \mathcal{L}(H^\rho_\Phi, H^{-\rho}_\Phi) \quad \forall t \right\}. \quad (6)$$

Note that (6) defines Fréchet spaces under the norms $\| a \|_t := \| a_t \|_{\mathcal{L}(H^\rho_\Phi, H^{-\rho}_\Phi)}$. In the case $\rho = 0$ it can be checked that (6) leads to a submultiplicative operator algebra.

By Corollary 1 the inclusion

$$\Psi^0_{1/2, 1/2} \subset \mathcal{A}_\Phi \subset \bigcap_{t \in \mathbb{R}} \mathcal{L}(H^1_\Phi, H^1_\Phi) \quad (7)$$

holds and for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ which are smooth in a neighbourhood of supp $\Phi$ the multiplication $M_f$ belongs to $\mathcal{A}_\Phi$. In such a sense $\mathcal{A}_\Phi$ can be seen as a localized version of $\Psi^0_{1/2, 1/2}$. To refine the algebras we can pose additional commutator conditions (prescribed properties): Let $U \subset \mathbb{R}^n$ be a bounded open set and assume that $\Phi \equiv 1$ on $U$. For finitely many vector fields

$$\mathcal{W} := \left\{ Y_1, \cdots, Y_k \right\} \subset \Psi^1_{1,0}$$

which are supported in $U$ consider the sub-algebra of $\mathcal{A}_\Phi$ defined by:

$$\Psi_{\mathcal{W}}[\mathcal{A}_\Phi] := \left\{ a \in \mathcal{A}_\Phi : \text{ad}[Y_{i_1}, \cdots, Y_{i_m}] (a) \in \bigcap_{s \in \mathbb{Z}} \mathcal{L}(H^s_\Phi, H^{s-\frac{m}{2}}_\Phi) \quad \text{for } Y_{i_t} \in \mathcal{W} \right\}. \quad (7)$$

Then $\mathcal{A}_\Phi$ in (7) can be replaced by any of the smaller algebras $\Psi_{\mathcal{W}}[\mathcal{A}_\Phi]$. Moreover, Theorem 2. The algebra $\Psi_{\mathcal{W}}[\mathcal{A}_\Phi]$ is spectrally invariant in $\mathcal{L}(L^2(\mathbb{R}^n))$. In case all vector fields $Y_j \in \mathcal{W}$ have real valued coefficients it defines a $\Psi^*$-algebra in $\mathcal{L}(L^2(\mathbb{R}^n))$.

4. Application to Toeplitz operators

For $n \in \mathbb{N}$ consider the Euclidean open unit ball $\mathbb{B}_{n+1} \subset \mathbb{C}^{n+1}$. The upper half space is defined by:

$$\mathcal{H}_+ := \left\{ (z, z_{n+1}) \in \mathbb{C}^{n+1} : z \in \mathbb{C}^n \text{ and } \text{Im } z_{n+1} > \| z \|^2 \right\}$$

where $\| \cdot \|$ denotes the usual Euclidean norm on $\mathbb{C}^n$. It is known - cf. [13] - that there is a biholomorphic function $F : \mathcal{H}_+ \to \mathbb{B}_{n+1}$ which induces a map of boundaries:

$$\tilde{F} : \partial \mathcal{H}_+ \to \partial \mathbb{B}_{n+1} \setminus \{ \text{south pole} \}.$$  

Consider the Heisenberg group $\mathbb{H}^{2n+1}$ of real dimension $2n + 1$. There is a simply transitive group homomorphism $g : \mathbb{H}^{2n+1} \to \text{Aut}(\mathcal{H}_+)$ which is induced by translations $g(x)$ along $x \in \mathbb{H}^{2n+1}$ such that the boundary $\partial \mathcal{H}_+$ is preserved under $g(x)$. Via the translation of $0 \in \partial \mathcal{H}_+$ one can identify:

$$\partial \mathcal{H}_+ \cong \mathbb{H}^{2n+1} \cong \mathbb{R}^{2n+1} \quad (8)$$
and we equip $\partial \mathcal{H}_+$ through (8) with the left invariant Haar measure $\beta$ on $\mathbb{H}^{2n+1}$. Here $\beta$ coincides with the $2n+1$-dimensional Lebesgue measure. The Hardy space $H^2(\partial \mathcal{H}_+)$ consists of all $f \in L^2(\partial \mathcal{H}_+,\beta)$ having holomorphic extensions to $\mathcal{H}_+$. Let

$$\tilde{P} : L^2(\partial \mathcal{H}_+,\beta) \rightarrow H^2(\partial \mathcal{H}_+)$$

be the orthogonal projection. Then $\tilde{P}$ corresponds via (8) to a (singular) convolution operator $P \in \mathcal{L}(L^2(\mathbb{H}^{2n+1}))$. In the framework of pseudo-differential operators $P$ can be described as follows:

**Theorem 3.** (A. Nagel, E.M. Stein, cf. [11]) Let $\varphi, \psi \in \mathcal{D}(\mathbb{H}^{2n+1})$, then $\varphi P \psi$ is a pseudo-differential operator of exotic type $\Psi^{0}_{\frac{1}{2},\frac{1}{2}}$.

Theorem 3 can be generalized to arbitrary strictly pseudo-convex domains instead of $\mathbb{B}_{n+1}$, cf. [11]. Let $h \in L^\infty(\mathbb{H}^{2n+1})$ be a bounded measurable function, then we call

$$T_h := PM_h \in \mathcal{L}(L^2(\mathbb{H}^{2n+1}))$$

a Toeplitz operator with symbol $h$. Let $U \subset \mathbb{H}^{2n+1}$ be the open set in the construction of the algebras $\Psi_W[A_\Phi]$ in section 3 and $U \subset V \subset \mathbb{H}^{2n+1}$ ($V$ open). According to Theorem 3 and our remarks before we have:

**Theorem 4.** Let both $f \in L^2(\mathbb{H}^{2n+1})$ and the symbol $h$ be smooth in $V$. Then the same is true for $T_h f$. Let $\varphi \in \mathcal{D}(\mathbb{H}^{2n+1})$ such that $\text{supp} \varphi \subset U$, then $\varphi T_h \in \Psi_W[A_\Phi]$.

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