SMOOTH RIGIDITY AT INFINITY OF THE UNIVERSAL COVER
OF CLOSED MANIFOLDS OF NEGATIVE CURVATURE

CHENGBO YUE

ABSTRACT. It has been conjectured that if the ideal boundary of the universal cover of a closed Riemannian manifold $(M,g)$ of negative sectional curvature has a $C^2$-structure which is invariant under the action of the fundamental group, or equivalently, if the horospherical foliation of the geodesic flow of such a manifold is $C^2$, then $(M,g)$ must be locally symmetric. This talk describes an approach to this problem.

1. INTRODUCTION

Let $(M,g)$ be a closed $(n+1)$-dimensional $(n \geq 1)$ Riemannian manifold of negative sectional curvature. Let $\widetilde{M}$ be its universal cover and let $\partial \widetilde{M}$ be the ideal boundary of $\widetilde{M}$. It is well-known that $\widetilde{M}$ carries a $C^\alpha$-Hölder structure $(0 < \alpha < 1)$ (see [AS] for example).

One way to look at this is as follows. For $x \in \widetilde{M}$, let $\pi_x : S_x \widetilde{M} \to \partial \widetilde{M}$, $v \mapsto v(\infty)$ be the projection of the unit tangent sphere to the ideal boundary. Since $\pi_x$ is one to one, one can push the natural topology of the unit tangent sphere onto $\partial \widetilde{M}$ to obtain a topology of the ideal boundary. The only problem is, if $y \in \widetilde{M}$ is another point, the topology on $\partial \widetilde{M}$ induced by $\pi_y : S_y \widetilde{M} \to \partial \widetilde{M}$ might not coincide with the topology induced by $\pi_x$. If one thinks of $\pi_x : S_x \widetilde{M} \to \partial \widetilde{M}$ as local coordinate charts and $\pi_y^{-1} \circ \pi_x$ as the coordinate change, then the problem is equivalent to whether these coordinate changes are continuous or smooth with respect to the canonical smooth structure of the unit tangent sphere.

Another way to look at this problem is from a dynamical point of view. Namely, since $M$ is negatively curved, the geodesic flow $g_t$ on the unit tangent bundle $SM$ is an Anosov flow, i.e., there exists $g_t$-invariant splitting, the so called Anosov splitting, $TSM = E^+ + E^- + E^0$, where $E^0$ is generated by the geodesic spray and $E^+$ (resp. $E^-$) is exponentially expanded (resp. contracted) by the geodesic flow. It turns out that the Anosov splitting of the geodesic flow is $C^k$ if and only if for all $x,y \in \widetilde{M}$, the map $\pi_y^{-1} \circ \pi_x : S_x \widetilde{M} \to S_y \widetilde{M}$ is $C^k$ with respect to the canonical smooth structure of the unit tangent sphere.

Anosov [A] proved that these subbundles $E^+, E^-$ are Hölder continuous. If $\dim M = 2$ or if the sectional curvature of $M$ is strictly quarter-pinched, i.e., $-4\lambda^2 < K < -\lambda^2$, then these subbundles are $C^1$ [Gre, HP]. However, except in the very special case of locally symmetric spaces of negative curvature, there are no other known examples for which these subbundles are $C^2$-differentiable. Indeed, it has been conjectured that the Anosov

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splitting of a closed Riemannian manifold \((M,g)\) of negative sectional curvature is \(C^2\) if and only if \((M,g)\) is locally symmetric (see for example [Kan], [DG]).

If \(\dim M = 2\), Ghys [Gh] proved that if the Anosov splitting is \(C^\infty\), then the geodesic flow of \(M\) is smoothly time-preserving conjugate to the geodesic flow of a hyperbolic surface. Hurder-Katok [HK] proved that if the Anosov splitting is \(C^2\), then it is \(C^\infty\). Combining with the result of Ghys, they proved that \(M\) itself must be hyperbolic.

In the higher dimensional case, Kanai [Kan] introduced one of the key tools in the study of smooth rigidity problems of Riemannian manifold of negative curvature, the so-called Kanai connection, and proved that if the Anosov splitting is \(C^\infty\) and if the sectional curvature of \(M\) is sufficiently pinched, then the geodesic flow of \(M\) is smoothly time-preserving conjugate to the geodesic flow of a hyperbolic manifold. After a series of works by Feres, Katok et al [FK1, FK2, F], Benoist-Foulon-Labourie [BFL], applying a general principle of Gromov [Gro] on rigid transformation groups, was eventually able to prove that there exists some large number \(k\) which depends on the dimension of \(M\) (see also [Gro]) such that if the Anosov splitting of \(M\) is \(C^k\), then the geodesic flow of \(M\) is smoothly time-preserving conjugate to the geodesic flow of a locally symmetric space.

In this talk I described an new approach to this conjecture.

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2. **Symplectic Structure and the Busemann Pairing Function**

Let \((M,g)\) be a closed \((n+1)\)-dimensional \((n \geq 1)\) Riemannian manifold of negative sectional curvature. Let \(\tilde{M}\) be its universal cover and let \(\partial \tilde{M}\) be the ideal boundary of \(\tilde{M}\). The fundamental group \(\Gamma = \pi_1 M\) acts continuously on the ideal boundary.

For \(x \in \tilde{M}\), let \(\pi_x : S_x \tilde{M} \to \partial \tilde{M}, v \mapsto v(\infty)\) be the projection of the unit tangent sphere to the ideal boundary. It is well-known that the Anosov splitting of the geodesic flow is \(C^k\) if and only if for all \(x, y \in \tilde{M}\), the map \(\pi_y^{-1} \circ \pi_x : S_x \tilde{M} \to S_y \tilde{M}\) is \(C^k\) with respect to the canonical smooth structure of the unit tangent sphere. One can think of \(\pi_x : S_x \tilde{M} \to \partial \tilde{M}\) as local coordinate charts and \(\pi_y^{-1} \circ \pi_x\) as the coordinate change. Thus if the Anosov splitting is \(C^k\), then the ideal boundary carries an natural \(C^k\)-structure and the fundamental group \(\Gamma\) acts by \(C^k\)-homeomorphisms.

Fix \(x \in \tilde{M}, \xi \in \partial \tilde{M}\). Let \(\gamma(t)\) be the unique unit speed geodesic such that \(\gamma(0) = x, \gamma(\infty) = \xi\), the Busemann function \(\rho_{x,\xi}(y)\) is defined to be

\[
\rho_{x,\xi}(y) = \lim_{t \to \infty} (t - d(y, \gamma(t))).
\]

Let \(G\tilde{M}\) be the space of geodesics. Then \(G\tilde{M}\) can be identified with \((\partial \tilde{M} \times \partial \tilde{M}) \setminus D\) where \(D\) is the diagonal. The smooth structure of \(S\tilde{M}\) descends to a smooth structure on \(G\tilde{M}\) which in turn induces a smooth structure on \((\partial \tilde{M} \times \partial \tilde{M}) \setminus D\). The stable and unstable foliations of the geodesic flow project to two foliations \(L^-, L^+\) on \(G\tilde{M} = (\partial \tilde{M} \times \partial \tilde{M}) \setminus D\) which coincides with its product structure, i.e., leaves of the \(L^-\) (resp. \(L^+\))-foliation are exactly the sets \((\partial \tilde{M} \setminus y) \times y\) (resp. \(x \times (\partial \tilde{M} \setminus x)\)). The action of the fundamental group \(\Gamma\) on \(\partial \tilde{M}\) extends to a diagonal action on \((\partial \tilde{M} \times \partial \tilde{M}) \setminus D\) by smooth diffeomorphisms.

We construct a Busemann pairing function \(B(\xi, \eta)\) which satisfies the following
Proposition 1. For any $\gamma \in \Gamma$, $B(\gamma \xi, \gamma \eta) = B(\xi, \eta) + \phi(\xi) + \psi(\eta)$.

Moreover, we prove that

Proposition 2. If the Anosov splitting of the geodesic flow is $C^k$, then for any $x, y \in \tilde{M}$, the Busemann pairing function $B(\xi, \eta)$ is $C^k$ in $\xi, \eta$.

Let $\sigma$ be the canonical contact form of the geodesic flow on $SM$. Then $\sigma(X^0) = 1$ where $X^0$ is the geodesic spray and $\sigma(Y) = 0$ for all $Y \in E^- \oplus E^+$. Since the exterior differential $\omega = d\sigma$ is invariant under the geodesic flow, it descends to a smooth symplectic form $\Omega$ on $(\partial \tilde{M} \times \partial \tilde{M}) \setminus D$. The foliations $L^-, L^+$ are Lagrangian with respect to $\Omega$. We denote their tangent bundle by $F^-, F^+$.

If the Anosov splitting of the geodesic flow is $C^2$. Then the Busemann pairing $B(\xi, \eta)$ is $C^2$ in $\xi, \eta$. We define a 2-form $\Theta$ as follows:

1. First define a 1-form $\theta$ by $\theta(X) = 0$ if $X \in F^-$ and $\theta(Y) = Y(B)$ if $Y \in F^+$;
2. The 2-form $\Theta$ is defined to be $\Theta = d\theta$.

It is easy to check that $\Theta(X, Y) = X(Y(B)) - \theta([X, Y])$ for $X \in F^-, Y \in F^+$ and $\Theta(X_1, X_2) = \Theta(Y_1, Y_2) = 0$ for $X_i \in F^-, Y_i \in F^+(i = 1, 2)$.

For any point $(\xi, \eta) \in (\partial \tilde{M} \times \partial \tilde{M}) \setminus D$, take $C^2$-local coordinates $(x_1, \ldots, x_n)$ around $\xi$ on $\partial \tilde{M} \setminus \eta$ and $C^2$-local coordinates $(y_1, \ldots, y_n)$ around $\eta$ on $\partial \tilde{M} \setminus \xi$, then $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ defines $C^2$-local coordinates around $(\xi, \eta)$ on $(\partial \tilde{M} \setminus \eta) \times (\partial \tilde{M} \setminus \xi) \setminus D$, the 1-form $\theta$ can be written as

$$\theta = \sum_{j=1}^{n} \frac{\partial B}{\partial y_j} dy_j.$$

Its exterior differential is given by $\Theta = d\theta = \sum_{i,j} \frac{\partial^2 B}{\partial x_i \partial y_j} dx_i \wedge dy_j$.

Proposition 3. The 2-form $\Theta$ is $\Gamma$-invariant.

Moreover, using the dynamics of the geodesic flow we can prove

Proposition 4. The 2-form $\Theta$ coincides with the canonical symplectic form $\Omega$.

Thus the pairing function satisfies a system of partial differential equations $\frac{\partial^2 B}{\partial x_i \partial y_j} = \omega_{ij}, 1 \leq i, j \leq n$. Using this, we can prove that if the Anosov splitting of the geodesic flow is $C^2$, then it is $C^\infty$. By [BFL], $M$ is isometric to $N$.

REFERENCES


(Chengbo Yue) Academy of Mathematical and System Sciences, Chinese Academy of Sciences

E-mail address: yue@amss.ac.cn