ROBIN FUNCTIONS FOR COMPLEX MANIFOLDS AND APPLICATIONS

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ABSTRACT. We give a variational formula of the Robin constant $\lambda(t)$ for a family of domains $D(t)$ in a complex manifold $M$ where $t$ is a complex parameter $t$. As an application we give a necessary and sufficient condition that a pseudoconvex domain with $C^\infty$ boundary in a complex homogeneous space fails to be a Stein domain. Using this condition we classify such domains in the complex flag space and in special Hopf manifolds. Proofs may be found in arXiv 0710.1091.

1. INTRODUCTION

In [8] we analyzed the second variation of the Robin function associated to a smooth variation of domains in $\mathbb{C}^n$ for $n \geq 2$; i.e., $D = \bigcup_{t \in B(t, D(t))} B \times \mathbb{C}^n$ is a variation of domains $D(t)$ in $\mathbb{C}^n$ each containing a fixed point $z_0$ and with $\partial D(t)$ of class $C^\infty$ for $t \in B := \{ t \in \mathbb{C} : |t| < \rho \}$. For such $t$ and for $z \in D(t)$ we let $g(t, z)$ be the $\mathbb{R}^{2n}$-Green function for the domain $D(t)$ with pole at $z_0$; i.e., $g(t, z)$ is harmonic in $D(t) \setminus \{z_0\}$, $g(t, z) = 0$ for $z \in \partial D(t)$, and $g(t, z) - \frac{1}{|z - z_0|^{2n-2}}$ is harmonic near $z_0$. We call

$$\lambda(t) := \lim_{z \to z_0} \left[ g(t, z) - \frac{1}{|z - z_0|^{2n-2}} \right]$$

the Robin constant for $(D(t), z_0)$. Then

$$\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z)|\nabla_z g|^2 dS_z - 4c_n \int_{D(t)} \sum_{a=1}^{n} \left| \frac{\partial^2 g}{\partial t \partial \overline{z}_a} \right|^2 dV_z. \quad (1)$$

Here, $c_n = \frac{1}{(n-1)! \Omega_n}$ is a positive dimensional constant where $\Omega_n$ is the area of the unit sphere in $\mathbb{C}^n$, $dS_z$ and $dV_z$ are the Euclidean area element on $\partial D(t)$ and volume element on $D(t)$, $\nabla_z g = (\frac{\partial g}{\partial z_1}, \ldots, \frac{\partial g}{\partial z_n})$ and

$$k_2(t, z) := ||\nabla_z \psi||^{-3} \left[ \frac{\partial^2 \psi}{\partial t \partial \overline{t}} ||\nabla_z \psi||^2 - 2\Re \left\{ \frac{\partial \psi}{\partial t} \sum_{a=1}^{n} \frac{\partial \psi}{\partial \overline{z}_a} \frac{\partial^2 \psi}{\partial \overline{z}_a} \right\} + \left| \frac{\partial \psi}{\partial t} \right|^2 \Delta_z \psi \right]$$

is the Levi-curvature of $\partial D$ at $(t, z)$. The function $\psi(t, z)$ is a defining function for $D$. In particular, if $D$ is pseudoconvex at a point $(t, z)$ with $z \in \partial D(t)$, it follows that $k_2(t, z) \geq 0$ so that $-\lambda(t)$ is subharmonic in $B$. Given a bounded domain $D$ in $\mathbb{C}^n$, we let $\Lambda(z)$ be the Robin constant for $(D, z)$. We call $\Lambda(z)$ the Robin function for $D$. Then the above formula yields part of the following result (cf., [8]).

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Theorem 1. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ boundary. Then $\log(-\Lambda(z))$ and $-\Lambda(z)$ are real-analytic, strictly plurisubharmonic exhaustion functions for $D$.

This theorem is striking in the the sense that, whereas in one complex variable, any harmonic function is locally the real part of a holomorphic function, in several complex variables ($\mathbb{C}^n$ with $n \geq 2$), harmonic functions are not closely related to holomorphic functions. On the other hand, it is known in the theory of functions of several complex variables that the plurisubharmonic functions $s(z)$, i.e., $[\frac{\partial^2 s}{\partial z_i \partial \overline{z}_j}]_{i,j} \geq 0$, have an intimate relation with holomorphic functions. The Robin function $\Lambda(\zeta)$ on $D$, which is constructed from a harmonic function, the Green function, has the property that $-\Lambda(\zeta)$ is a plurisubharmonic function on $D$. Thus it is important to generalize the variational formula (1) to complex manifolds (see [9]).

We now study a generalization of the second variation formula (1) to complex manifolds $M$ equipped with a Hermitian metric $ds^2$ and a smooth, nonnegative function $c$. Our purpose is that, with this added flexibility, we are able to give a criterion for a bounded, smoothly bounded, pseudoconvex domain $D$ in a complex homogeneous space to be Stein (Theorem 5). In particular, we are able to do the following:

1. Describe concretely all the non-Stein pseudoconvex domains $D$ in the complex torus of Grauert (section 5).
2. Give a description of all the non-Stein pseudoconvex domains $D$ in the special Hopf manifolds $\mathbb{H}_n$ (section 7).
3. Give a description of all the non-Stein pseudoconvex domains $D$ in the complex flag spaces $\mathcal{F}_n$ (section 8).

The metric $ds^2$ and the function $c$ give rise to a $c$-Green function and $c$-Robin constant associated to an open set $D \subset M$ and a point $p_0 \in D$. We then take a variation $D = \bigcup_{t \in B} \{D(t) \subset B \times M$ of domains $D(t)$ in $M$ each containing a fixed point $p_0$ and define a $c$-Robin function $\lambda(t)$. The precise definitions of these notions and the new variation formula (2) will be given in the next section. In section 3 we impose a natural condition (see (3)) on the metric $ds^2$ which will be useful for applications. Kähler metrics, in particular, satisfy (3). After discussing conditions which insure that the function $-\lambda$ is subharmonic, we will use (2) to develop a "rigidity lemma" (Lemma 1) which will imply, if $-\lambda$ is not strictly subharmonic, the existence of a nonvanishing, holomorphic vector field on $M$ with certain properties (Corollary 2). This will be a key tool to obtain the purpose stated above.

The majority of this paper is the same as in [7] with minor changes.

2. The variation formula.

Our general set-up is this: let $M$ be an $n$-dimensional complex manifold (compact or not) equipped with a Hermitian metric $ds^2 = \sum_{a,b=1}^{n} g_{ab} dz_a \otimes d\overline{z}_b$ and let $\omega := i \sum_{a,b=1}^{n} g_{ab} dz_a \wedge d\overline{z}_b$ be the associated real $(1,1)$ form. As in the introduction, we take $n \geq 2$. We write $g^{\overline{a}b} := (g_{ab})^{-1}$ for the elements of the inverse matrix to $(g_{ab})$ and
$G := \det(g_{ab})$. We denote by $\Delta$ the Laplacian associated to $ds^2$,

$$
\Delta u = -2\left[ \sum_{a,b=1}^{n} g^{\bar{a}a} \frac{\partial^2 u}{\partial \bar{z}_b \partial z_a} + \frac{1}{2} \sum_{a,b=1}^{n} \left( \frac{1}{G} \frac{\partial (Gg^{\bar{a}a})}{\partial z_a} \frac{\partial u}{\partial \bar{z}_b} + \frac{1}{G} \frac{\partial (Gg^{\bar{a}b})}{\partial \bar{z}_a} \frac{\partial u}{\partial z_b} \right) \right].
$$

Given a nonnegative $C^\infty$ function $c = c(z)$ on $M$, we call a $C^\infty$ function $u$ on an open set $D \subset M$ c-harmonic on $D$ if $\Delta u + cu = 0$ on $D$. Choosing local coordinates near a fixed point $p_0 \in M$ and a coordinate neighborhood $U$ of $p_0$ such that $[g_{ab}(p_0)]_{a,b=1,...,n} = [\delta_{ab}]_{a,b=1,...,n}$, the Laplacian $\Delta$ corresponds to a second-order elliptic operator $\Delta$ in $\mathbb{C}^n$. In particular, we can find a c-harmonic function $Q_0$ in $U \setminus \{p_0\}$ satisfying

$$
\lim_{p \to p_0} Q_0(p)d(p,p_0)^{2n-2} = 1
$$

where $d(p,p_0)$ is the geodesic distance between $p$ and $p_0$ with respect to the metric $ds^2$. We call $Q_0$ a fundamental solution for $\Delta$ and $c$ at $p_0$. Fixing $p_0$ in a smoothly bounded domain $D \subset M$ and fixing a fundamental solution $Q_0$, the c-Green function $g$ for $(D,p_0)$ is the c-harmonic function in $D \setminus \{p_0\}$ satisfying $g = 0$ on $\partial D$ (is continuous up to $\partial D$) with $g(p) - Q_0(p)$ regular at $p_0$. The c-Green function always and uniquely exists (cf., [10]) and is nonnegative on $D$. Then

$$
\lambda := \lim_{p \to p_0} [g(p) - Q_0(p)]
$$

is called the c-Robin constant for $(D,p_0)$. Thus we have

$$
g(p) = Q_0(p) + \lambda + h(p)
$$

for $p$ near $p_0$, where $h(p_0) = 0$. In case $M$ is compact, if $c \equiv 0$ on $M$, then the c-Green function $g$ for $(M,p_0)$ exists and is positive on $M$, hence the c-Robin constant is finite. But if $c \equiv 0$ on $M$, a c-harmonic function is harmonic and cannot attain its minimum; thus, in this case, $g(z) = +\infty$ on $M$ (cf., [10]). In this case we set $\lambda = +\infty$.

Now let $D = \cup_{t \in B} (t, D(t)) \subset B \times M$ be a $C^\infty$ variation of domains $D(t)$ in $M$ each containing a fixed point $p_0$ and with $\partial D(t)$ of class $C^\infty$ for $t \in B$. This means that there exists $\psi(t,z)$ which is $C^\infty$ in a neighborhood $N \subset B \times M$ of $\{t,z\} : t \in B, z \in \partial D(t)$, negative in $N \cap \{(t,z) : t \in B, z \in D(t)\}$, and for each $t \in B, z \in \partial D(t)$, we require that $\psi(t,z) = 0$ and $\frac{\partial \psi}{\partial z_i}(t,z) \neq 0$ for some $i = 1, ..., n$. We call $\psi(t,z)$ a defining function for $D$. Assume that $B \times \{p_0\} \subset D$. Let $g(t,z)$ be the c-Green function for $(D(t),p_0)$ and $\lambda(t)$ the corresponding c-Robin constant. The hypothesis that $D$ be a $C^\infty$ variation implies that for each $t \in B$, the c-Green function $g(t,z)$ extends of class $C^\infty$ beyond $\partial D(t)$; this follows from the general theory of partial differential equations.

Our formulas are the following:

$$
\frac{\partial \lambda}{\partial t}(t) = -c_n \int_{\partial D(t)} k_1(t,z) \sum_{a,b=1}^{n} (g^{\bar{a}b} \frac{\partial g}{\partial z_b} \frac{\partial g}{\partial z_a})d\sigma_z,
$$

$$
\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = -c_n \int_{\partial D(t)} k_2(t,z) \sum_{a,b=1}^{n} (g^{\bar{a}b} \frac{\partial g}{\partial \bar{z}_a} \frac{\partial g}{\partial \bar{z}_b})d\sigma_z
$$

$$
- \frac{c_n}{2n-2} \left\{ \left( \frac{\partial g}{\partial t} \right)^2 \bigg|_{D(t)} + \frac{1}{2} \left( \sqrt{c} \frac{\partial g}{\partial t} \right)^2 \bigg|_{D(t)} + \frac{1}{2} \Re \int_{D(0)} \frac{\partial g}{\partial t} \left[ \frac{1}{i} \partial * \omega \wedge \frac{\partial g}{\partial t} + \frac{1}{i} \frac{\partial g}{\partial \bar{t}} \right] \right\},
$$

(2)
where $d\sigma_z$ is the area element on $\partial D(t)$ with respect to the Hermitian metric and

$$k_1(t, z) := \left[ \sum_{a, b=1}^n g^{ab} \frac{\partial \psi}{\partial z_a} \frac{\partial \psi}{\partial z_b} \right]^{-1/2} \frac{\partial \psi}{\partial t},$$

$$k_2(t, z) := \left[ \sum_{a, b=1}^n g^{ab} \frac{\partial \psi}{\partial z_a} \frac{\partial \psi}{\partial z_b} \right]^{-3/2} \times$$

$$\left[ \frac{\partial^2 \psi}{\partial t \partial \overline{t}} \left( \sum_{a, b=1}^n g^{ab} \frac{\partial \psi}{\partial z_a} \frac{\partial \psi}{\partial z_b} \right) - 2 \Re \left\{ \frac{\partial \psi}{\partial t} \left( \sum_{a, b=1}^n g^{ab} \frac{\partial \psi}{\partial z_a} \frac{\partial^2 \psi}{\partial z_b \partial \overline{t}} \right) \right\} - \frac{1}{2} \left| \frac{\partial \psi}{\partial t} \right|^2 \Delta_z \psi \right],$$

$\psi(t, z)$ being a defining function for $D$. Here, $k_i(t, z)$ ($i = 1, 2$) is a real-valued function for $(t, z) \in \partial D$ which is independent of both the choice of defining function for $D$ and of the choice of local parameter $z$ in the manifold $M$. We call $k_2(t, z)$ the Levi scalar curvature with respect to the metric $ds^2$.

3. Subharmonicity of $-\lambda$.

We impose the following condition on the Hermitian metric $ds^2$ on $M$:

$$\partial \star \omega = 0 \quad \text{on} \quad M. \quad (3)$$

A Kähler metric $ds^2$ on $M$ (i.e., $d\omega = 0$) satisfies (3). Further we note that if $D$ is pseudoconvex in $B \times M$, then $k_2(t, z) \geq 0$ on $\partial D$. Thus by the variational formula (2) we have

**Theorem 2.** Assume that $ds^2$ satisfies condition (3) and $D$ is pseudoconvex in $B \times M$. Then:

$$\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t) \leq -\frac{c_n}{2^{n-1}} ||\nabla^c \frac{\partial g}{\partial t}||^2_{D(t)}.$$

Hence $-\lambda(t)$ is subharmonic on $B$.

4. Rigidity.

We continue under the same hypotheses: $M$ is an $n$-dimensional complex manifold equipped with a Hermitian metric $ds^2$; $D = \cup_{t \in B}(t, D(t)) \subset B \times M$ is a $C^\infty$ variation of domains $D(t)$ in $M$ each containing a fixed point $p_0$ and with $\partial D(t)$ of class $C^\infty$ for $t \in B$; and $c$ is a nonnegative $C^\infty$ function on $M$.

Throughout this section we will assume that

1. $ds^2$ satisfies condition (3) on $M$;
2. $D$ is pseudoconvex in $B \times M$.
3. $c(z) > 0$ on $M$.

By Theorem 2 we have

**Lemma 1** (Rigidity). If there exists $t_0 \in B$ at which $\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t_0) = 0$, then $\frac{\partial g}{\partial t}(t_0, z) \equiv 0$ for $z \in D(t_0)$.

**Corollary 1** (Trivial variation). If $\lambda(t)$ is harmonic in $B$, then $D = B \times D(0)$. Namely, the domain $D(t)$ does not move in $M$ with $t \in B$. 
We next consider the following set-up. Let $F : B \times M \to M$ be a holomorphically varying, one-parameter family of automorphisms of $M$; i.e., $F(t, z)$ is holomorphic in $(t, z)$ with $\frac{\partial F}{\partial t} \neq 0$ for $(t, z) \in B \times M$, and, for each $t \in B$, the mapping $F_t : M \to M$ via $F_t(z) := F(t, z)$ is an automorphism of $M$. Then the mapping $T : B \times M \to B \times M$ defined as

$$T(t, z) = (t, w) := (t, F(t, z))$$

provides a fiber-wise automorphism of $M$; i.e., for each $t \in B$, the map $w = F(t, z)$ is an automorphism of $M$.

We put $z = F^{-1}(t, w) = \Phi(t, w) = (\phi_1(t, w), \ldots, \phi_n(t, w))$. Then for each $t \in B$,

$$\Theta_t(z) := \sum_{k=1}^n \frac{\partial \phi_k}{\partial t}(t, F(t, z)) \frac{\partial}{\partial z_k}.$$ 

becomes a non-vanishing holomorphic vector field on $M$.

Fix a pseudoconvex domain $D \Subset M$ and let $\mathcal{D} := T(B \times D)$. This is a variation of pseudoconvex domains $D(t) = F(t, D)$ in the image space $B \times M$ of $T$. Assume there exists a common point $w_0$ in each domain $D(t)$, $t \in B$. Let $g(t, w)$ and $\lambda(t)$ denote the $c$-Green function and $c$-Robin constant of $(D(t), w_0)$. We obtain the following fundamental result, utilizing the rigidity lemma, Lemma 1.

**Corollary 2.**

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$$

at some $t_0 \in B$ if and only if $\Theta_{t_0}(z)$ is tangential on $\partial D$; i.e., the entire integral curve $I(z_0)$ associated to $\Theta$ for any initial point $z_0 \in \partial D$ lies on $\partial D$.

5. **Complex Lie Groups.**

We apply Corollary 2 to the study of complex Lie groups. Let $M$ be a complex Lie group of complex dimension $n$ with identity $e$. We always take $M$ to be connected. By [6], there always exists a Kähler metric on $M$; thus we conclude that $M$ is equipped with a Hermitian metric $ds^2$ satisfying condition (3). We fix such a Hermitian metric and a strictly positive $C^\infty$ function $c = c(z)$ on $M$ throughout this section. We denote by $\mathfrak{X}$ the complex Lie algebra which consists of all left-invariant holomorphic vector fields on $M$.

Let $D \Subset M$ be a pseudoconvex domain in $M$ with $C^\infty$ boundary. Fix $X \in \mathfrak{X} \setminus \{0\}$; $\zeta \in D$ and $B = \{t \in \mathbb{C} : |t| < \rho\}$ with $\rho > 0$ sufficiently small so that $\zeta \exp tX \in D$ for $t \in B$. We consider the holomorphic map $T : B \times M \to B \times M$ defined as

$$T(t, z) = (t, w) := (t, F(t, z)) \quad \text{where} \quad F(t, z) := z(\zeta \exp tX)^{-1}.$$ 

As in Corollary 2, we write $z = \Phi(t, w)$ if $w = F(t, z) = z(\zeta \exp tX)^{-1}$ so that $\Phi(t, w) = w(\zeta \exp tX)$. Let $\mathcal{D} = T(B \times D)$. Fixing a value of $t \in B$, we will write

$$D(t) := F(t, D) = D \cdot (\zeta \exp tX)^{-1}.$$ 

Note for $t \in B$ fixed, $F(t, \cdot) \in \text{Aut}(M)$. Furthermore, since we have $e \in D(t)$ for each $t \in B$, we can thus construct the $c$-Robin constant $\lambda(t)$ for $(D(t), e)$; and we have the following.
Lemma 2. Suppose $\frac{\partial^2 \lambda}{\partial t^2}(t_0) = 0$ for some $t_0 \in B$. Then the integral curve $z \exp tX, t \in \mathbb{C}$, with initial value $z$, of the holomorphic vector field $X$ satisfies, for $A = D, \partial D$, or $\overline{D}$,

1. $z \in A$ implies $\{z \exp tX : t \in \mathbb{C}\} \subseteq A$;
2. $A \cdot z^{-1} = A \cdot (z \exp tX)^{-1}$ for all $t \in \mathbb{C}$ and all $z \in M$.

We make another observation based on Lemma 2. Consider the following automorphism $T$ of $M \times M$: $T(z, w) = (z, \lambda) := (z, wz^{-1})$. Let $D \subseteq M$ be a domain with $C^\infty$ boundary and let

$$D := T(D \times D) = \bigcup_{z \in D} (z, D(z))$$

where

$$D(z) := D \cdot z^{-1} = \{wz^{-1} : w \in D\}.$$ 

This is a variation of domains $D(z)$ in $M$ with parameter space $D \subset M$. Note that $e \in D(z)$ for all $z \in D$. Let $G(z, W)$ be the c-Green function for $(D(z), e)$ and $\Lambda(z)$ the c-Robin constant. Then $\Lambda(z)$ is a $C^\infty$ function on $D$, called the c-\textbf{Robin function} for $D$.

We use Lemma 2 to prove the following result, which will be crucial in all that follows.

Lemma 3. Suppose $D$ is pseudoconvex. Then $-\Lambda(z)$ is a plurisubharmonic exhaustion function for $D$. Furthermore, assume that $-\Lambda(z)$ is not strictly plurisubharmonic at some point $z_0$ in $D$. We put

$$\mathfrak{X}_{z_0} = \{X \in \mathfrak{X} : \{z_0 \exp tX : t \in \mathbb{C}\} \subseteq D\}.$$ 

Then

1. $\mathfrak{X}_{z_0}$ is a complex Lie subalgebra in $\mathfrak{X}$ with $1 \leq \dim \mathfrak{X}_{z_0} \leq n - 1$; Thus we have the integral manifold (i.e., the corresponding connected Lie subgroup) $\Sigma_{z_0}$ in $M$ passing through $z_0$;
2. For each $z \in D$ we have $z\Sigma_{z_0} \subseteq D$.

We remark that the conclusion of the lemma implies, in particular, that $D$ is not Stein. We now address the following question: for a complex Lie group $M$, when is a pseudoconvex domain $D \subseteq M$ with $C^\infty$ boundary Stein? An answer is provided in the following result.

Theorem 3. Let $D \subseteq M$ be a pseudoconvex domain with smooth boundary which is not Stein. Then

I. there exists a unique connected complex Lie subgroup $H$ of $M$ such that

1. $1 \leq \dim H \leq n - 1$;
2. $D$ is foliated by cosets of $H$; $D = \bigcup_{z \in D} zH$ with $zH \subseteq D$;
3. any holomorphic curve $\ell := \{z = z(t) \in D : t \in \mathbb{C}\}$ with $\ell \in D$ is necessarily contained in some coset $zH$ in $D$.

II. Furthermore, if $H$ is closed in $M$ and $\pi : M \rightarrow M/H$ is the canonical projection, then $H$ is a complex torus and there exists a Stein domain $D_0 \subseteq M/H$ with smooth boundary such that $D = \pi^{-1}(D_0)$.

Remark 1. In Theorem 3 if $H$ is not closed in $M$ the closure $H$ of $H$ in $M$ is a closed real Lie subgroup of $M$ whose real dimension $m$ is less than $2n$. In this case, we have the real Lie subalgebra $\mathfrak{r}_0 \subset \mathfrak{X}$ corresponding to $H$, and the projection $\pi : M \rightarrow M/H,$
where $M/H$ is a real manifold of dimension $2n - m$. From properties 1. and 2. in Lemma 2, $H$ is a compact real submanifold in $D$ and $D$ is foliated by right cosets of $H$. Furthermore, $D_0 := \pi(D)$ is a relatively compact domain with smooth boundary in $M/H$ and $D = \pi^{-1}(D_0)$.

We next discuss a concrete example of complex Lie groups $M$ as in Theorem 3 or Remark 1. Grauert gave an example of a complex Lie group $M$ and a pseudoconvex domain $D \subset M$ with smooth boundary which is not Stein. Moreover, in his example, $D$ admits no nonconstant holomorphic functions. This domain lies in a complex torus $T$ of complex dimension 2. Our next goal is to describe all pseudoconvex subdomains $D$ of $T$ with smooth boundary which are not Stein (cf., O. Suzuki [13], T. Ohsawa [12]). The key tools we will use are Theorem 3 and Remark 1. We begin with real 4-dimensional Euclidean space $\mathbb{R}^4$ with coordinates $x = (x_1, x_2, x_3, x_4)$. Let

$$e_1 = (1, 0, 0, 0), \ e_2 = (0, 1, 0, 0), \ e_3 = (0, 0, 1, 0), \ e_4 = (0, 0, 0, 0)$$

in $\mathbb{R}^4$,

where $\xi$ is an irrational number. Initially we consider the real 4-dimensional torus: $T := \mathbb{R}^4/[e_1, e_2, e_3, e_4] = T_1 \times T_2$, where $T_1 = \mathbb{R}[x_1, x_2]/[(1, 0), (0, 1)]$ and $T_2 = \mathbb{R}[x_3, x_4]/[(1, 0), (0, \xi)]$. Following Grauert, we impose the complex structure $z = x_1 + ix_3, \ w = x_2 + ix_4$ on $T$. Then $T$, equipped with this complex structure, becomes a complex torus $T$ of complex dimension 2. Note that $e_1, e_2, e_3, e_4$ correspond to $(1, 0), (0, 1), (i, 0), (i\xi, i)$ in $\mathbb{C}^2$, and the complex Lie algebra of the complex Lie group $T$ is $\mathfrak{t} = \{\alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial w} : \alpha, \beta \in \mathbb{C}\}$. Grauert showed that $D = D(c_1, c_2) := \{c_1 < \Re z < c_2 \} \subset T$, where $0 \leq c_1 < c_2 < 1$, is a pseudoconvex domain which admits no nonconstant holomorphic functions.

Let $D \subset T$ be a pseudoconvex domain with smooth boundary which is not Stein. We consider the $c$-Robin function $\Lambda(z, w)$ on $D$, where $c \equiv 1$ on $T$. By Theorem 3 there exists $X = \alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial w} \in \mathfrak{t}$ with $(\alpha, \beta) \neq (0, 0)$ such that $D(\exp tX)^{-1} = D, \ t \in \mathbb{C}$. Since the integral curve $\exp tX, \ t \in \mathbb{C}$ for $X$ passing through 0 in $T$ is the curve $(z, w) = (at, \beta t) \in \mathbb{C}_2 \times \mathbb{C}_w, \ t \in \mathbb{C}$, the equality $D(\exp tX)^{-1} = D$ for $t \in \mathbb{C}$ simply means that $D + (at, \beta t) = D$ for $t \in \mathbb{C}$. Since dim $T = 2$, the Lie subalgebra $\mathfrak{t}_0$ associated to $D$ from Theorem 3 and the corresponding Lie subgroup $H$ are of the form

$$\mathfrak{t}_0 = \{cX \in \mathfrak{t} : c \in \mathbb{C}\}, \ H = \{(at, \beta t) \in T : t \in \mathbb{C}\}.$$ 

We have three cases: (1) $\alpha = 0; \ (2) \ \beta = 0; \ (3) \ \alpha, \ \beta \neq 0$. In the cases (1) and (2) it is not difficult to determine the form of the domain $D$. The specific example of Grauert is case (1). In case (3) we write $\beta/\alpha = a + ib$ where $a$, $b$ are real. If $b = 0$ it is again straightforward to determine the form of $D$. The situation when $b \neq 0$ is the most general and most interesting case to determine the form of $D$. In this case the integral curve $\exp tX, \ t \in \mathbb{C}$ starting at $(0, 0)$ can be written in the form

$$S : x_2 + ix_4 = (a + ib)(x_1 + ix_3) \quad \text{in } \mathbb{R}^4. \quad (4)$$

Then we have the following:

1. (i) There exist six integers $m, n, m', n' \in \mathbb{Z}, n', p, q \in \mathbb{Z}^+$ where $(m, n) = \pm 1, (m', n') = \pm 1, (p, q) = 1$, such that $a, b$ can be written in the following form:

$$a = \frac{p_1 p_2 + q_1 q_2}{p_1^2 + q_1^2}, \quad b = \frac{p_2 q_1 - p_1 q_2}{p_1^2 + q_1^2}, \quad (5)$$
where
\[ M' := m' + n' \xi, \quad p_1 := M'p, \quad p_2 := n'p, \quad q_1 := mq, \quad q_2 := nq. \] (6)

1. (ii) The integral curve \( S \) in (4) contains the following two points:
\[ (q(m, n), p(M', n')), \quad (p(1/n, 0), q(1/n', 0) + \eta(M', n')) \]
where \( \eta = \frac{p_1^2 + q_2^2}{p_2q_1 - p_1q_2} \) is irrational.

2. (i) The closure \( \bar{H} \) of \( H \) in \( T \) is a closed real Lie subgroup of \( T \) whose corresponding real Lie subalgebra \( \mathfrak{t}_0 \) of \( X \) is generated by
\[ \left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{p_1}{x_1} \frac{\partial}{\partial x_3} + p_2 \frac{\partial}{\partial x_4}, \right. \\
\left. \frac{p_2q_1 - p_1q_2}{x_1} \frac{\partial}{\partial x_3} + (p_1p_2 + q_1q_2) \frac{\partial}{\partial x_3} + (p_2^2 + q_2^2) \frac{\partial}{\partial x_4} \right\}. \]

We proceed to give a more precise description of \( \bar{H} \). Assuming 1., let
\[ L_1 : \{ (x_1, x_2) : mx_2 = nx_1 \} = \{ t(m, n) : t \in \mathbb{R} \} \text{ in } \mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; \]
\[ L_2 : \{ (x_3, x_4) : M'x_4 = n'x_3 \} = \{ t(M', n') : t \in \mathbb{R} \} \text{ in } \mathbb{R}_{x_3} \times \mathbb{R}_{x_4}. \]

Since \( m, n \in \mathbb{Z} \), \( L_1 \) defines a simple closed curve \( l_1 \) in the real torus \( T_1 \), and from (6), specifically, from the relation \( M' = m' + n' \xi \), \( L_2 \) defines a simple closed curve \( l_2 \) in the real torus \( T_2 \). Given \( 0 \leq s \leq 1 \), define
\[ L_1(s) := L_1 + ps(1/n, 0) = \{ t(m, n) + ps(1/n, 0) : t \in \mathbb{R} \} \text{ in } \mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; \]
\[ L_2(s) := L_2 + qs(1/n', 0) = \{ t(M', n') + qs(1/n', 0) : t \in \mathbb{R} \} \text{ in } \mathbb{R}_{x_3} \times \mathbb{R}_{x_4}. \]

Then \( L_1(s) \) and \( L_2(s) \) also define simple closed curves \( l_1(s) \) and \( l_2(s) \) in \( T_1 \) and \( T_2 \); \( l_1(s) \) is a curve in \( T_1 \), which, in \( \mathbb{R}_{x_1} \times \mathbb{R}_{x_2} \), is parallel to \( l_1 \) translated by the vector \( ps(1/n, 0) \). Similarly, \( l_2(s) \) is parallel to \( l_2 \) in \( \mathbb{R}_{x_3} \times \mathbb{R}_{x_4} \) translated by the vector \( qs(1/n', 0) \). We have \( l_i = l_i(0) = l_i(1) \) for \( i = 1, 2 \) and
\[ (l_1(s') \times l_2(s')) \cap (l_1(s'') \times l_2(s'')) = \emptyset \text{ if } s' \neq s''. \]

2. (ii) The set \( \Sigma := \bigcup_{0 \leq s \leq 1} l_1(s) \times l_2(s) \) is a real, 3-dimensional compact submanifold of \( T \), and \( \bar{H} = \Sigma \). Given \( 0 \leq t \leq 1 \), if we define \( \Sigma(t) := (t, 0; 0, 0) + \bar{H} = (t, 0; 0, 0) + \Sigma \), a coset of \( \bar{H} \), then \( \Sigma(0) = \Sigma(1) = \Sigma = \bar{H} \) and \( \Sigma(t') \cap \Sigma(t'') = \emptyset \) in \( T \) if \( t' \neq t'' \).

3. We have \( T/\bar{H} = \mathbb{R}/[1] = S^1 \) and \( D = \bigcup_{0 \leq t_1 < t_2 \leq 1} \Sigma(t) \), where \( 0 \leq t_1 < t_2 < 1 \).

We also have a converse statement.

4. Given integers \( m, n, m' \in \mathbb{Z}; \ n', p, q \in \mathbb{Z}^+ \) with \( (m, n) = \pm 1 \), \( (m', n') = \pm 1 \), \( (p, q) = 1 \), we can find \( a, b \in \mathbb{R} \) satisfying (5) and (6) to construct a pseudoconvex domain \( D \subset T \) with smooth boundary which is not Stein. This domain has the property that \( D(p \exp tX)^{-1} = D \) for all \( t \in \mathbb{C} \) and for all \( p \in D \) where \( X \) is a nonzero holomorphic vector field. The Lie subgroup \( H \) of \( T \) corresponding to the Lie subalgebra \( \mathfrak{X}_0 = \{ cX \in X : c \in \mathbb{C} \} \) is equal to \( \{ w = (a + bi)z \} \). Moreover, every holomorphic function on \( D \) is constant.
Remark 2. Let $\ell_1^*$ be a conjugate closed curve for $\ell_1$ in $T_1$, i.e., take $m_1, n_1 \in \mathbb{Z}$ such that $m_1 n - mn_1 = 1$; thus the vertices $(0, 0), (m, n)$ and $(m_1, n_1)$ determine a fundamental domain of $T_1$. The curve $\ell_1^*$ corresponds to the segment joining $(0, 0)$ and $(m_1, n_1)$ in $T_1$. Similarly, let $\ell_2^*$ be a conjugate closed curve for $\ell_2$ in $T_2$ determined by the relation $m_1 n' - m'n_1 = 1$. The following figure gives a visual interpretation of the Lie subgroup $H$ of $\mathfrak{T}$ and its closure $\overline{H}$. The set $T_1^{p,q}$ is the $pq$-sheeted torus over $T_1$ winding $p$ times along $\ell_1^*$ and $q$ times along $l_1$, while $T_2^{q,p}$ is the $pq$-sheeted torus over $T_2$ winding $q$ times along $\ell_2^*$ and $p$ times along $l_2$.

$$ F(qm + pm_1, qn + pn_1) $$


In this section, we let $M$ be an $n$-dimensional complex space with the property that there exists a connected complex Lie group $G \subset \text{Aut} M$ of complex dimension $m \geq n$ which acts transitively on $M$. We write $e$ for the identity element of $G$ and $\mathfrak{X}$ for the Lie algebra which consists of all left-invariant holomorphic vector fields on $G$. For a fixed $z \in M$, let

$$ H_z := \{ g \in G : g(z) = z \} $$

be the isotropy subgroup of $G$ for $z$. We let $G/H_z$ denote the set of all left cosets $gH_z$. This quotient space $G/H_z$ has the structure of a complex $n$-dimensional manifold such that if we let $\pi_z : G \to G/H_z$ be the coset mapping $\pi_z(g) = gH_z$ and we let $\psi_z : G \to M$ be the mapping $\psi_z(g) = g(z)$, then there exists an isomorphism $\alpha_z : G/H_z \to M$ such that $\alpha_z \circ \pi_z = \psi_z$ in $G$.

Let $D \subset M$ be a domain with $C^\infty$ boundary in $M$. For $z \in D$, we let

$$ D(z) := \{ g \in G : g(z) \in D \} = \psi_z^{-1}(D). $$

so that $D(z)$ contains $e$ and is a possibly unbounded and possibly disconnected domain in $G$. Thus $D(z)$ is a set of cosets modulo $H_z$ in $G$. 

We set $H'_z(D'(z))$ the connected component of $H_z(D(z))$ which contains $e$ so that $D'(z) \supset H'_z$. We also set $H'(z) = D'(z) \cap H_z$. Then:

**Proposition 1.** We have, for $z \in D$,

1. $D = \{g(z) \in M : g \in D'(z)\}$.
2. $D'(z)h \subset D'(z)$ for $h \in H'(z)$.
3. $D'(h(z)) = D'(z)h^{-1}$ for $h \in D'(z)$.
4. $H'(z)$ is a closed Lie subgroup of $H_z$.

We define $D' := \cup_{z \in D} (z, D'(z)) \subset D \times G$. This is a variation of domains $D'(z) \subset G$ with parameter $z \in D$:

$$D' : z \in D \rightarrow D'(z) \subset G.$$

**Lemma 4.** $D'$ is locally holomorphically trivial variations.

Fix a Kähler metric $ds^2$ on $G$ (such a metric exists by [6]) and let $c$ be a strictly positive $C^\infty$ function on $G$. We consider the c-Robin constant $\lambda(z)$ for $(D'(z), e)$ in case when $D'(z)$ is not bounded by the usual exhaustion method in the case of an unbounded connected domain $D(z)$ (see, for example, Chapter IV in [2]). Using standard methods from potential theory, from Lemma 4 we see that $\lambda(z)$ is smooth in $D$. Furthermore, since $\partial D$ is smooth in $G$ and since $z \in D \rightarrow \zeta \in \partial D$ implies $\partial D'(z) \rightarrow e$, we have $\lambda(z) \rightarrow -\infty$ as $z \rightarrow \partial D$.

Then we have

**Theorem 4.** If $D \Subset M$ is a smoothly bounded pseudoconvex domain, then the c-Robin function $-\lambda(z)$ for $D$ is a plurisubharmonic exhaustion function for $D$.

We next discuss conditions under which $-\lambda(z)$ is strictly plurisubharmonic on $D$. Suppose not; i.e., suppose there exists a point $z_0 \in D$ at which the complex Hessian

$$\left[\frac{\partial^2 (-\lambda)}{\partial z_j \partial \bar{z}_k}(z_0)\right]$$

has a zero eigenvalue so that $\left[\frac{\partial^2 (-\lambda(z_0+at))}{\partial t \partial \bar{t}}\right]_{t=0} = 0$ for some direction $a \in \mathbb{C}^n$. 
(a \neq 0). By a standard result in the theory of homogeneous spaces we find an \( X \) in \( \mathfrak{X} \) such that the tangent vector of \( \exp tX(z_0) \) at \( t = 0 \) in \( M \) is equal to \( a \); and thus \( \left[ \frac{\partial^2 \lambda(\exp tX(z_0))}{\partial t \partial t} \right]_{t=0} = 0. \)

Note that \( \exp tX \in D'(z_0) \) for \( |t| < \rho \) if \( \rho \) is sufficiently small. It follows from 3. in Proposition 1 that

\[
D'(\exp tX(z_0)) = D'(z_0) \cdot (\exp tX)^{-1}.
\]

We thus apply Lemma 2 to \( G, D'(z_0) \), and \( \zeta \) corresponding to \( M, D \), and \( \zeta \) in the lemma — note that the domain \( D \) in the lemma is bounded, but the argument is valid for unbounded \( D \) — to immediately obtain the following facts: if we assume \( g \in D'(z_0) \), then

a. the integral curve \( \{ \exp tX : t \in \mathbb{C} \} \) for \( X \) is contained in \( D'(z_0) \).

b. \( D'(z_0) \cdot g^{-1} = D'(z_0) \cdot (\exp tX)^{-1} \) for all \( t \in \mathbb{C} \); i.e., \( D'(g(z_0)) = D'(g \exp tX(z_0)) \) for all \( t \in \mathbb{C} \).

c. \( \lambda(g(z_0)) = \lambda(g \exp tX(z_0)) \) for all \( t \in \mathbb{C} \).

d. \( \{ g \exp tX(z_0) : t \in \mathbb{C} \} \) is relatively compact in \( D \).

We consider the following subsets of \( \mathfrak{X} \):

\[
g_{z_0} := \text{the Lie subalgebra of } G \text{ corresponding to } H'_{z_0};
\]

\[
\mathfrak{X}_{z_0} := \{ X \in \mathfrak{X} : X \text{ satisfies } \left[ \frac{\partial^2 \lambda(\exp tX(z_0))}{\partial t \partial t} \right]_{t=0} = 0 \}.
\]

By using b. inductively we have: For \( \nu \in \mathbb{Z}^+ \) and \( g_i, h_i \in H'_{z_0} \); \( X_i \in \mathfrak{X}_{z_0} \); \( t_i \in \mathbb{C}, i = 1, 2, \ldots, \nu \) and \( g \in D'(z_0) \), it holds

(i) \( g \left[ \prod_{i=1}^{\nu} g_i(\exp t_i X_i)h_i^{-1} \right] \in D'(z_0) \);

(ii) \( \lambda(g \left[ \prod_{i=1}^{\nu} (g_i(\exp t_i X_i)h_i^{-1}) \right](z_0)) = \lambda(g(z_0)) \).

This formulas imply the following fundamental result.

**Lemma 5.**

1. \( \mathfrak{X}_{z_0} \) is a complex Lie subalgebra of \( \mathfrak{X} \) such that \( g_{z_0} \subseteq \mathfrak{X}_{z_0} \subseteq \mathfrak{X} \).
2. Let \( X \in \mathfrak{X}_{z_0} \) and \( g \in H'_{z_0} \). Then any \( Y \in \mathfrak{X} \) which satisfies

\[
\left[ \frac{d X Y(z_0)}{dt} \right]_{t=0} = \left[ \frac{d g \exp tX(z_0)}{dt} \right]_{t=0}
\]

belongs to \( \mathfrak{X}_{z_0} \).

3. For \( z_1 \in D \) we have \( \dim \mathfrak{X}_{z_1} = \dim \mathfrak{X}_{z_0} \).

We write \( \Sigma_{z_0} \) for the integral manifold for \( \mathfrak{X}_{z_0} \) in \( G \) and put \( \sigma_{z_0} = \psi_{z_0}(\Sigma_{z_0}) \subset M \); i.e.,

\[
\Sigma_{z_0} = \{ \prod_{j=1}^{\nu} \exp t_j A_j : \nu \in \mathbb{Z}^+, t_j \in \mathbb{C}, A_j \in \mathfrak{Y} \cup \mathfrak{Z} \}
\]

\[
\sigma_{z_0} = \Sigma_{z_0}(z_0) = \{ g(z_0) \in M : g \in \Sigma_{z_0} \}.
\]

We thus have \( \Sigma_{z_0} \subset D'(z_0) \) and \( \sigma_{z_0} \subset D \). By Frobenius theorem the Lie group \( G \) is foliated by \( g\Sigma_{z_0}, g \in G \).

To state the main theorem we introduce the following terminology. Let \( \sigma \) be a subset of \( M \). We call \( \sigma \) a \( \mu \)-dimensional non-singular, generalized (f-generalized) analytic set in \( M \) if, for any point \( z \in \sigma \) (\( z \in M \)), there exists a neighborhood \( V \) of \( z \) in \( M \) such that each connected component of \( V \cap \sigma \) is a \( \mu \)-dimensional non-singular analytic set in
V. Moreover, if \( \sigma \) admits no non-constant bounded plurisubharmonic functions, then \( \sigma \) is said to be parabolic. Further, for \( A, B \subset G \), we set \( AB = \{ ab \in G : a \in A, b \in B \} \).

Then we have the main theorem in this paper:

**Theorem 5.** Let \( M \) be a complex homogeneous space of finite dimension \( n \) and let \( G \) be a connected complex Lie transformation group of finite dimension \( m \geq n \) which acts transitively on \( M \). Let \( D \) be a pseudoconvex domain in \( M \) with smooth boundary. Assume that \( D \) is not Stein. Fix \( z_0 \in D \) and consider \( \Sigma_{z_0} \) defined by (7). Then:

a. The subset \( \sigma_{z_0} = \Sigma_{z_0}(z_0) \) of \( M \) is a parabolic \( m_1 \)-dimensional (where \( 1 \leq m_1 \leq n \)) non-singular \( f \)-generalized analytic set in \( M \) passing through \( z_0 \) with the following properties:

(o) \( \sigma_{z_0} \subset D; \Sigma_{z_0}H'(z_0) = D'(z_0) \cap \text{Aut} \sigma_{z_0}; \Sigma_{z_0}H'(z_0) \) is a Lie subgroup of \( G \) which acts transitively on \( \sigma_{z_0} \); and

\[ \sigma_{z_0} \approx \Sigma_{z_0}H'(z_0)/H'(z_0). \]

(i) There exists a domain \( K \) with \( D \subset K \subset M \) such that \( D(K) \) is foliated by the sets \( g\sigma_{z_0} \), \( g \in K'(z_0) \) \((g \in D'(z_0))\) and each such set \( g\sigma_{z_0} \) is relatively compact in \( K(D) \):

\[ K = \bigcup_{g \in K'(z_0)} g\sigma_{z_0} \quad \text{and} \quad D = \bigcup_{g \in D'(z_0)} g\sigma_{z_0}. \]

(ii) Any dimensional parabolic non-singular generalized analytic set \( \sigma \) in \( M \) which is relatively compact in \( D \) is contained in a set \( g\sigma_{z_0} \) for some \( g \in D(z_0) \).

b. Assume that \( \sigma_{z_0} \) is closed in \( M \). Then

(o) \( \sigma_{z_0} \) is a compact manifold containd in \( D; \Sigma_{z_0}H'(z_0) = D'(z_0) \cap \text{Aut} \sigma_{z_0}; \Sigma_{z_0}H'(z_0) \) is a Lie subgroup of \( G \) which acts transitively on \( \sigma_{z_0} \); and

\[ \sigma_{z_0} \approx \Sigma_{z_0}H'(z_0)/H'(z_0). \]

(i) There exists a complex manifold \( K_0 \) and a holomorphic map \( \pi_0 : K \hookrightarrow K_0 \) such that \( \pi_0^{-1}(\zeta) \approx \sigma_{z_0} \) (as complex manifolds) for each \( \zeta \in K_0 \); moreover, there exists a Stein domain \( D_0 \subset K_0 \) with smooth boundary such that \( D = \pi_0^{-1}(D_0) \).

c. In the case when \( H_{z_0} \) is connected in \( G \) or more generally in the case when \( H_{z_0} \subset \Sigma_{z_0} \), we can take \( K = M \) in a. and b. In particular, in this situation, if, in addition \( \sigma_{z_0} \) is closed in \( M \) (i.e., b. holds), we have \( K_0 = G/\Sigma_{z_0} \) and \( \pi_0 : M = G/H_{z_0} \hookrightarrow K_0 = G/\Sigma_{z_0} \).

We note that the converse is clearly true; i.e., if a pseudoconvex domain \( D \) in \( M \) with \( C^\infty \) boundary satisfies (o) in a., then \( D \) is not Stein.

7. Special Hopf manifolds

The second special case in c. described in Theorem 5 occurs when \( M \) is a special Hopf manifold. We let \( (\mathbb{C}^n)^* := \mathbb{C}^n \setminus \{0\} \) and we fix \( \alpha \in \mathbb{C} \) with \( |\alpha| \neq 1 \). For \( z, z' \in (\mathbb{C}^n)^* \), we define the following equivalence relation in \( (\mathbb{C}^n)^* \): \( z \sim w \) iff \( w = \alpha^k z \) for some integer \( k \). We consider the following space:

\[ H_n = (\mathbb{C}^n)^*/\sim, \]
this is an \(n\)-dimensional compact manifold. The space \(\mathbb{H}_n\) is called a special Hopf mani-
fold. We write \([z] \in \mathbb{H}_n\) to denote the equivalence class of a point \(z = (z_1, \ldots, z_n) \in (\mathbb{C}^n)^*\). In case \(n = 1\), \(\mathbb{H}_1\) is the usual one-dimensional complex torus \(T_\alpha\). The space \(\mathbb{H}_n\) clearly has the following property:

**Proposition 2.**

1. The group \(GL(n, \mathbb{C})\) is a Lie transformation group of \(\mathbb{H}_n\) and acts transitively on \(\mathbb{H}_n\); i.e., \(\mathbb{H}_n\), equipped with the Lie group \(GL(n, \mathbb{C})\), is a homogeneous space.
2. We have the canonical projection \(\pi_0 : [z_1, \ldots, z_n] \in \mathbb{H}_n \mapsto [z_1 : \ldots : z_n] \in \mathbb{P}^{n-1}\) such that \(\pi_0^{-1}(\zeta) \approx T_\alpha\) for each \(\zeta \in \mathbb{P}^{n-1}\).

We write \(O := [(1, 0, \ldots, 0)] \in \mathbb{H}_n\) and call \(O\) the base point of \(\mathbb{H}_n\). Then the isotropy subgroup \(H_0\) of \(GL(n, \mathbb{C})\) for \(O\) is

\[
H_0 = \left\{ \begin{pmatrix} \alpha^k & (*) \\ \vdots & A \\ 0 \end{pmatrix} \in GL(n, \mathbb{C}) : k \in \mathbb{Z}, (\ast) \in \mathbb{C}^{n-1}, \det A \neq 0 \right\},
\]

which is closed but not connected in \(GL(n, \mathbb{C})\). We let \(H'_0\) denote the connected com-
ponent of \(H_0\) which contains the identity \(I_n\) in \(GL(n, \mathbb{C})\), i.e.,

\[
H'_0 = \left\{ \begin{pmatrix} 1 & (*) \\ \vdots & A \\ 0 \end{pmatrix} \in GL(n, \mathbb{C}) : (\ast) \in \mathbb{C}^{n-1}, \det A \neq 0 \right\}.
\]

Therefore, \(\mathbb{H}_n \approx GL(n, \mathbb{C})/H_0 = \{gH_0 : g \in GL(n, \mathbb{C})\}\). To be precise, let \(g = (g_{ij})_{i,j} \in GL(n, \mathbb{C})\). Let \(g := (g_{11}, \ldots, g_{n1}) \in (\mathbb{C}^n)^*\) denote the first column vector of \(g\). Then the mapping

\[
\alpha_0 : gH_0 \in GL(n, \mathbb{C})/H_0 \to g(O) = [g] \in \mathbb{H}_n
\]

is bijective. For local coordinates in a neighborhood \(V\) of the base point \(O\) we can take

\[
\begin{pmatrix}
1 + t_1 \\
t_2 & 1 \\
\vdots & \ddots \\
t_n & \end{pmatrix}, \quad |t_i| < \rho, \quad i = 1, \ldots, n,
\]

where the missing entries are all 0. Equivalently, let \(U := \{t = (t_1, \ldots, t_n) \in \mathbb{C}^n : |t_i| < \rho\}\) where the base point \(O\) of \(\mathbb{H}_n\) corresponds to the origin 0 of \(\mathbb{C}^n\). That is, let \(g = (g_{ij})_{ij} \in GL(n, \mathbb{C})\) be close to the identity \(I_n\). Corresponding to \(gH_0 \in \mathbb{H}_n\) the point \(t(g) := (g_{11} - 1, g_{21}, \ldots, g_{n1}) \in \mathbb{C}^n\), is close to \((0, 0, \ldots, 0)\). We have (i) if \(g_1H_0 \neq g_2H_0\) for \(g_1, g_2 \in V\), then \(t(g_1) \neq t(g_2)\); (ii) given \(t' \in U\), we can find \(g \in GL(n, \mathbb{C})\) close to \(I_n\) with \(t(g) = t'\). We call the local coordinates \(t\) at \(O\) the standard local coordinates at \(O\) in \(\mathbb{H}_n\).

We consider the Lie algebra \(\mathfrak{x}\) consisting of all left-invariant holomorphic vector fields \(X\) on \(GL(n, \mathbb{C})\). We identify \(\mathfrak{x}\) with \(M_n(\mathbb{C})\), the set of all \(n \times n\) square matrices, as
follows: to \( X = (\lambda_{ij}) \in M_n(\mathbb{C}) \) we associate a left-invariant holomorphic vector field \( v_X \) on \( GL(n, \mathbb{C}) \) via, for \( g = (x_{ij}) \in GL(n, \mathbb{C}) \),

\[
  v_X(g) := \sum_{i,j=1}^n \lambda_{ij} X_{ij}(g), \quad \text{where} \quad X_{ij}(g) = \sum_{k=1}^n x_{ki} \frac{\partial}{\partial x_{kj}}.
\]  

(8)

Hence we identify the vector field \( v_X \) on \( GL(n, \mathbb{C}) \) with the matrix \( X = (\lambda_{ij}) \) in \( M_n(\mathbb{C}) \) as additive groups. The integral curve \( C_X \) for \( v_X \) with initial value \( I_n \) is given by

\[
  C_X = \{ \exp tX \in GL(n, \mathbb{C}) : t \in \mathbb{C} \},
\]

and the integral curve of \( v_X \) with initial value \( g \in GL(n, \mathbb{C}) \) is given by \( g C_X \in GL(n, \mathbb{C}) \).

We let \( g_0 \) denote the corresponding Lie subalgebra for \( H_0' \), so that

\[
  g_0 = \left\{ \begin{pmatrix} 0 & \vdots & A \\ 0 & \vvdots & \vdots \\ \vdots & \vdots & \end{pmatrix} \in M_n(\mathbb{C}) : A \in M_{n,n-1}(\mathbb{C}) \right\},
\]

where \( M_{n,n-1}(\mathbb{C}) \) denotes the set of all \( n \times (n-1) \)-matrices. We have

\[
  H_0' = \{ \prod_{i=1}^\nu t_i X_i \in GL(n, \mathbb{C}) : \nu \in \mathbb{Z}^+, \ t_i \in \mathbb{C}, \ X_i \in g_0 \}.
\]

Then we have

**Theorem 6.** Let \( D \subseteq \mathbb{H}_n \) be a pseudoconvex domain with smooth boundary which is not Stein. Then there exists a Stein domain \( D_0 \) in \( \mathbb{P}^{n-1} \) with smooth boundary such that \( D = \pi_0^{-1}(D_0) \).

**Proof.** We may assume \( D \) contains the base point \( O \). Following Theorem 5 for such domains \( D \) in a homogeneous space, we fix a Kähler metric \( ds^2 \) on \( G = GL(n, \mathbb{C}) \) and a strictly positive \( C^\infty \) function \( c \) on \( G \) and we consider the \( c \)-Robin function \( \lambda([z]) \) for \( D \).

Define the following subset of the Lie algebra \( \mathfrak{x} = M_n(\mathbb{C}) \):

\[
  \mathfrak{x}_0 = \{ X \in \mathfrak{x} : \frac{\partial^2 \lambda(\exp tX(O))}{\partial t \partial \bar{t}}|_{t=0} = 0 \}.
\]

Under our assumptions for \( D \) we showed that \( \mathfrak{x}_0 \) is a Lie subalgebra of \( \mathfrak{x} \) with

\[
  g_0 \subseteq \mathfrak{x}_0 \subseteq \mathfrak{x}.
\]

By the elementary calculus such Lie subalgebra \( \mathfrak{x}_0 \) satisfying (9) must be

\[
  \mathfrak{x}_0 = \left\{ \begin{pmatrix} x \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} : x \in \mathbb{C}, \ (\ast) \in M_{n,n-1}(\mathbb{C}) \right\},
\]

This implies the connected Lie subgroup \( \Sigma_0 \) correspondint to \( \mathfrak{x}_0 \) is:

\[
  \Sigma_0 = \left\{ \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} : a \in \mathbb{C}^*; \ (\ast) \in \mathbb{C}^{n-1}, \ A \in GL(n-1, \mathbb{C}) \right\},
\]
so that $\Sigma_0$ is closed in $GL(n, \mathbb{C})$ with $H_0 \subset \Sigma_0$, and hence $\Sigma_0$ is in case c. in Theorem 5. Moreover, writing $tO$ for the transpose of the row vector $O = [(1, 0, \ldots, 0)]$.

$$
\sigma_0 := \psi_0(\Sigma_0) = \Sigma_0(tO) = \{ (a, 0, \ldots, 0) : a \in \mathbb{C}^* \},
$$

which coincides with the torus $T_n$. Since $GL(n, \mathbb{C})/\Sigma_0 = \mathbb{P}^{n-1}$ and the projection $gH_0 \mapsto g\Sigma_{z_0}$ from $\mathbb{H}_n$ to $\mathbb{P}^{n-1}$ coincides with $\pi_0$ in 2. of Proposition 2, Theorem 6 follows from c. in Theorem 5. $\square$

8. Flag space

In this section, we apply the first case in c. in Theorem 5 to the flag space $\mathcal{F}_n$ to determine all pseudoconvex domains $D \subset \mathcal{F}_n$ with smooth boundary which are not Stein (see Theorem 7). There are few known results on the Levi problem in $\mathcal{F}_n$ (cf., Y-T. Siu [11], A. Hirschowitz [3], K. Adachi [1]) so far. By definition, the flag space $\mathcal{F}_n$ is the set of all nested sequences

$$
z : \{0\} \subset F_1 \subset \ldots \subset F_{n-1} \subset \mathbb{C}^n, \quad (10)
$$

where $F_i, \; i = 1, \ldots, n - 1$ is an $i$-dimensional vector subspace of $\mathbb{C}^n$. We describe the structure of $\mathcal{F}_n$ as a homogeneous space. Given $A = (a_{ij}) \in GL(n, \mathbb{C})$ we shall define an isomorphism $A$ of $\mathcal{F}_n$. Consider the linear transformation of $\mathbb{C}^n$ given by

$$
A : \begin{pmatrix}
Z_1 \\
\vdots \\
Z_n
\end{pmatrix} = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \ddots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix} \begin{pmatrix}
z_1 \\
\vdots \\
z_n
\end{pmatrix}.
$$

For $z \in \mathcal{F}_n$ as in (10), we then define $A(z) \in \mathcal{F}_n$ via

$$
A(z) : \{0\} \subset A(F_1) \subset A(F_2) \subset A(F_{n-1}) \subset \mathbb{C}^n.
$$

In this way $GL(n, \mathbb{C})$ acts transitively on $\mathcal{F}_n$; i.e., $\mathcal{F}_n$ is a homogeneous space with Lie transformation group $GL(n, \mathbb{C})$. We fix the following point $O$ in $\mathcal{F}_n$:

$$
O : \{0\} \subset F_1^0 \subset F_2^0 \subset \cdots \subset F_{n-1}^0 \subset \mathbb{C}^n,
$$

where

$$
F_i^0 : z_{i+1} = \cdots = z_n = 0, \quad i = 1, \ldots, n - 1.
$$

We call $O$ the base point of $\mathcal{F}_n$. The isotropy subgroup $H_0$ of $GL(n, \mathbb{C})$ for the point $O$ is the set of all upper triangular non-singular matrices:

$$
H_0 = \left\{ \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix} \in GL(n, \mathbb{C}) \right\}.
$$

In particular, $H_0$ is connected in $GL(n, \mathbb{C})$ and $\dim H_0 = \frac{n(n+1)}{2}$. Since $\mathcal{F}_n \cong GL(n, \mathbb{C})/H_0$, the flag space $\mathcal{F}_n$ is a compact manifold with $\dim \mathcal{F}_n = N := \frac{n(n-1)}{2}$. By the identification (8) of the Lie algebra $\mathfrak{g}$ consisting of all left-invariant holomorphic vector fields
on $GL(n, \mathbb{C})$ and the set $M_n(\mathbb{C})$ of all $n$ square matrices $n$ square, the Lie subalgebra $\mathfrak{g}_0$ which corresponds to $H_0$ is

$$ \mathfrak{g}_0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{C}) \right\}. $$

We consider the generalized flag spaces $\mathcal{F}^\mathfrak{m}_n$. Let

$$ \mathfrak{m} := (m_1, \ldots, m_\mu) $$

be a fixed, finite sequence of positive integers with $1 \leq m_j \leq n$ and $m_1 + \cdots + m_\mu = n$. Set $n_j := m_1 + m_2 + \cdots + m_j$ for $j = 1, \ldots, \mu$, and consider a nested sequence $\zeta$ of vector spaces in $\mathbb{C}^n$

$$ \zeta : \{0\} \subset S_{n_1} \subset S_{n_2} \subset \cdots \subset S_{n_{\mu-1}} \subset \mathbb{C}^n, $$

where $S_{n_j}$, $j = 1, \ldots, \mu - 1$, is an $n_j$-dimensional vector space. Let $\mathcal{F}^\mathfrak{m}_n$ denote the set of all such sequences $\zeta$. We call $\mathcal{F}^\mathfrak{m}_n$ the $\mathfrak{m}$-flag space in $\mathbb{C}^n$. In particular, $\mathcal{F}^\mathfrak{m}_n$ coincides with $\mathcal{F}_n$ if $\mu = n$. Clearly $GL(n, \mathbb{C})$ acts transitively on $\mathcal{F}^\mathfrak{m}_n$. We fix the following point in $\mathcal{F}^\mathfrak{m}_n$ as the base point:

$$ O^\mathfrak{m} : \quad S_{n_j} = \{ z_{n_{j+1}} = \cdots = z_n = 0 \}, \quad j = 1, \ldots, \mu - 1. $$

Thus the isotropy subgroup $H_0^\mathfrak{m}$ of $GL(n, \mathbb{C})$ for $O^\mathfrak{m}$ is the set of all matrices

$$ \begin{pmatrix} h_1 & (*) & (*) \\ 0 & h_j & (*) \\ 0 & 0 & h_\mu \end{pmatrix} $$

where $h_j \in GL(m_j, \mathbb{C})$, $j = 1, \ldots, \mu$. Hence

(i) $\mathcal{F}^\mathfrak{m}_n \cong GL(n, \mathbb{C})/H_0^\mathfrak{m}$;

(ii) $H_0 \subset H_0^\mathfrak{m}$ and $H_0^\mathfrak{m}/H_0 \cong \mathcal{F}_{m_1} \times \cdots \times \mathcal{F}_{m_\mu}$, where $\mathcal{F}_{m_j}$ is the usual flag space in $\mathbb{C}^{m_j}$;

(iii) there exists a holomorphic projection $\pi^\mathfrak{m} : gH_0 \in \mathcal{F}_n \mapsto gH_0^\mathfrak{m} \in \mathcal{F}^\mathfrak{m}_n$, such that

$$(\pi^\mathfrak{m})^{-1}(\zeta) \approx \mathcal{F}_{m_1} \times \cdots \times \mathcal{F}_{m_\mu}, \quad \zeta \in \mathcal{F}^\mathfrak{m}_n. $$

Then we have

**Theorem 7.** Let $D \Subset \mathcal{F}_n$ be a pseudoconvex domain with smooth boundary which is not Stein. Then there exists a unique sequence $\mathfrak{m} = (m_1, \ldots, m_\mu)$ with $1 < \mu < n$ and a Stein domain $D_0 \Subset \mathcal{F}^\mathfrak{m}_n$ with smooth boundary such that $D = (\pi^\mathfrak{m})^{-1}(D_0)$. 
Proof. It suffices to prove the theorem under the assumption that $D$ contains the base point $O$ in $\mathcal{F}_n$. We consider the $c$-Robin function $\Lambda(z)$ for $D$ and define

$$\mathcal{X}_0 := \{X \in M_n(\mathbb{C}) : \left[ \frac{\partial^2 \Lambda(\exp tX(O))}{\partial t \partial t} \right]_{t=0} = 0 \}.$$ 

We see from the assumption for $D$ and Lemma 5 $\mathcal{X}_0$ is a Lie subalgebra of $GL(n, \mathbb{C})$ such that $\mathfrak{g}_0 \subset \mathcal{X}_0 \subset M_n(\mathbb{C})$. By the elementary calculus with property (2) in Lemma 5 we see that such $\mathcal{X}_0$ must be the following form: there exists $\mathfrak{M} = (m_1, \ldots, m_\mu)$ with $1 < \mu < n$ such that $\mathcal{X}_0$ is the subset $H_0^{(\mathfrak{m})}$ of $M_n(\mathbb{C})$ which consists of all matrices of the form

$$
\begin{pmatrix}
  h_{m_1} & (*) & (*) \\
  0 & h_{m_j} & (*) \\
  0 & 0 & h_{m_\mu}
\end{pmatrix},
$$

where $h_{m_j} \in M_{m_j}(\mathbb{C})$, $j = 1, \ldots, \mu$, and each (*) is an arbitrary element in the corresponding space $M_{m_j,m_k}(\mathbb{C})$ (here $m_j < m_k$). It follows that the integral manifold $\Sigma_0$ of the Lie subalgebra $\mathcal{X}_0$ passing through $O$ in $GL(n, \mathbb{C})$ is $H_0^{(\mathfrak{m})} \cap GL(n, \mathbb{C})$ and hence it is equal to the isotropy subgroup $H_0^{(\mathfrak{m})}$ of $GL(n, \mathbb{C})$ at the identity $I$ for the generalized flag space $\mathcal{F}_n^{(\mathfrak{m})}$. Thus, for the flag space $\mathcal{F}_n$, the space $M_0 := GL(n, \mathbb{C})/\Sigma_0$, which was considered in c. in Theorem 5 for general homogeneous spaces, coincides with the space $\mathcal{F}_n^{(\mathfrak{m})}$. Consequently, the projection $\pi_0$ and the analytic set $\sigma \subset D \subset M$ defined in c. in Theorem 5 coincide with the projection $\pi_0^{(\mathfrak{m})} : AH_0 \in \mathcal{F}_n \rightarrow A\Sigma_0 = AH_0^{(\mathfrak{m})} \in \mathcal{F}_n^{(\mathfrak{m})}$ (where $A \in GL(n, \mathbb{C})$) and the analytic set $H_0^{(\mathfrak{m})}/H_0 \cong \bigcap_{j=1}^{\mu} \mathcal{F}_{m_j}$ for $\zeta \in \mathcal{F}_n^{(\mathfrak{m})}$. Using c. in Theorem 5, there exists a Stein domain $D_0 \subset \mathcal{F}_n^{(\mathfrak{m})}$ with smooth boundary such that $D = \pi_0^{-1}(D_0)$. Theorem 7 is completely proved.

The following remark is from T. Ueda. Consider two generalized flag spaces $\mathcal{F}_n^{(\mathfrak{m})}$ and $\mathcal{F}_n^{(\mathfrak{L})}$ in $\mathbb{C}^n$, where $\mathfrak{M} = (m_1, \ldots, m_\mu)$, $\mathfrak{L} = (l_1, \ldots, l_\nu)$, $\mu > \nu$, and

$$l_1 = m_1 + m_2 + \ldots + m_{j_1}, \ldots, l_\nu = m_{j_{\nu-1}+1} + \ldots + m_\mu.$$

We introduce the notation $\mathfrak{M} < \mathfrak{L}$ for this situation. Then we have the canonical projection

$$\pi^{(\mathfrak{m})}_{\mathfrak{L}} : gH_0^{(\mathfrak{m})} \in \mathcal{F}_n^{(\mathfrak{m})} \mapsto gH_0^{(\mathfrak{L})} \in \mathcal{F}_n^{(\mathfrak{L})},$$

where $H_0^{(\mathfrak{m})}$ is the isotropy subgroup of $G$ for $\mathcal{F}_n^{(\mathfrak{m})}$ at the base point $O^{(\mathfrak{m})}$. Thus, for each $z \in \mathcal{F}_n^{(\mathfrak{L})}$,

$$(\pi^{(\mathfrak{m})}_{\mathfrak{L}})^{-1}(z) \approx \mathcal{F}_n^{(\mathfrak{m})}_{l_1} \times \ldots \times \mathcal{F}_n^{(\mathfrak{m})}_{l_\nu}$$

as complex manifolds.

where $\mathfrak{M}_k = (m_{j_{k-1}+1}, \ldots, m_{j_k})$, $k = 1, 2, \ldots, \nu$. If $\mathfrak{M} = (1, \ldots, 1)$, i.e., $\mathcal{F}_n^{(\mathfrak{m})} = \mathcal{F}_n$, we simply write $\pi^{(\mathfrak{m})}_{\mathfrak{L}} = \pi_{\mathfrak{L}}$. 
By use of Theorem 7 we have:

**Corollary 3.** Let $D$ be a pseudoconvex domain with smooth boundary in $F_n^{\mathbb{R}}$ which is not Stein. Then there exists a unique $\mathcal{L}$ such that $\mathfrak{M} \prec \mathcal{L}$ and a Stein domain $D_0$ in $F_n^{\mathbb{C}}$ with smooth boundary such that $D = (\pi_\mathcal{L})^{-1}(D_0) = D$.

T. Ueda has another proof of this corollary following ideas in the paper [1] (which is based on [14]).

**REFERENCES**


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